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CALCULATION AND ANALYSIS OF AIRCRAFT MOTION

by S. A. Gorbatenko, E. M. Makashov, Yu. F. Polushkin, and L. V. Sheftel

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16. Abstract

Methods of calculating and analyzing motion, trajectories, transient processes, control and stability of aircraft are described. Calculation and properties of various guidance methods are discussed. Control based on linear automatic control theory, nonlinear mechanics and theory of stability of nonlinear systems are described. Problems of optimal control theory and methods of solution are explained. Numerical methods of calculating trajectories, transient processes, steady state flight and control optimization problems are presented.

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ANNOTATION

This manual describes methods used for calculating and analyzing the motion of aircraft: their trajectories, transient processes, control and stability. Included are methods of calculation and properties of different methods of guidance to the target, methods of their study as an object of control based on the linear theory of automatic control, nonlinear mechanics and the theory of stability of nonlinear systems. The principal problems of the theory of optimal control as applied to aircraft and methods for their solution are also examined. Numerical methods for calculation of trajectories, transient processes, and steady state flight conditions are presented as well as numerical methods for solving problems of control optimization. Examples which explain the technique of using different methods for calculation and analysis are given.

All of the factual material on aircraft and their properties are hypothetical.

The manual is intended for scientific workers and engineers engaged in flight mechanics and investigation of aircraft control systems. It may also be useful for students in the corresponding educational institutions.

Translation Editor's Note: The Russian term, letatel'nyy apparat, has been consistently translated as "aircraft" throughout this book. This is its narrow meaning which applies particularly to flight within the limits of the earth's atmosphere. In its broad meaning, letatel'nyy apparat is properly translated as "flying vehicle". The reader should bear this distinction in mind when reading this book.

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PREFACE

Engineering methods of the calculation and analysis of aircraft motion are continuously being developed and improved. On the one hand this has been encouraged by the appearance of new problems for aircraft and the complication of ones that are already known, and on the other hand by the practical implementation of planning for modern scientific achievements and methods of the theory of optimal processes, nonlinear mechanics, the theory of automatic control, and numerical methods for computer problem solving.

In examining modern methods for investigation of problems of aircraft flight mechanics, a tendency toward their further differentiation into methods of synthesis (planning) of motions, which have given characteristics, and into methods of analysis of completely described movements can be seen. It should be pointed out, by the way, that the boundary between these methods is sometimes quite arbitrary. Guidance methods, methods of optimal control theory, methods for synthesis of automatic control theory may be considered methods of the first group; methods of the second group include methods of nonlinear mechanics, methods for analyzing automatic control theory and certain numerical solution methods. In this manual both groups of methods are examined with emphasis placed on guidance trajectories, aircraft motion stability, trajectories and methods of control, which optimize certain quality criteria and also on motion around a center of mass.

In addition, the manual presents numerical methods for solving equations for many problems of flight mechanics. The selection of specific numerical methods for solving algebraic, transcendental and differential equations was made with regard to accuracy and efficiency of their use in computer calculations.

This manual is a logical extension of the previous work of the authors — the manual "Flight Mechanics[1];" used herein are the conventional designations, coordinate systems and notations of flight mechanics which were used in the first manual. If the first manual made it possible to describe aircraft motion mathematically, then the second provides methods for calculating and analyzing this motion and methods for synthesis of a motion which has the necessary technical properties (for example, which satisfies specific boundary conditions or optimality according to any given quality criterion).

The manual consists of four chapters.

In Chapter 1 different guidance methods are examined and solutions of kinematic equations of motion which describe these methods are presented. Solutions are given for cases where guidance equations are written in two coordinate systems — Cartesian and relative polar.

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[1]See item [7].

Chapter 2 is devoted to methods of investigating aircraft as a controlled object (linear or nonlinear). The methods presented here enable one to evaluate the stability and quality of aircraft motion in a trajectory and around its center of mass.

In Chapter 3 different formulations of problems of optimum control or optimum selection of planned aircraft parameters which minimize or maximize certain aircraft quality criteria are presented. Formulations of necessary conditions of control optimality based on the maximum principle, dynamic programming and classical variational calculus are presented.

Chapter 4 illustrates methods of numerical integration of motion equations, methods for calculating equilibrium flight conditions, solutions of boundary problems, the numerical methods for optimization of controlling functions which correspond to different problems of flight mechanics.

The authors have attempted to present the material in the manual in a form which is most convenient for practical use. Description of the majority of methods is accompanied by standard examples which illustrate the sequence of operations in engineering calculations.

In order to find the desired material in the manual one should turn to the subject index and the quite detailed table of contents. The principal symbols used in each chapter are listed at its beginning and the remaining symbols are given in the text.

The authors wish to thank the Doctor of Technical Sciences, Professor I. V. Strazheva and Associate Professor N. N. Zavidonov for valuable comments regarding the manuscript.

Please direct all comments about the book to: Moscow, B-66, 1-Y Basmannyy per., 3, "Mashinostroyeniye" Press.

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CALCULATION AND ANALYSIS OF AIRCRAFT MOTION

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CHAPTER 1. GUIDANCE METHODS

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The problem of guidance consists in placing one aircraft (the interceptor) into a definite position relative to another aircraft (the target). In doing so not only the relative coordinates of the interceptor but also the relative velocities and flight course can be assigned at the end of the vectoring segment.

Differential and finite relations between the coordinates (Cartesian or curvilinear), linear and angular velocities or other kinematic parameters of the flight of the interceptor and target, which determine the relative motion of the interceptor during guidance execution, are called *guidance methods*.

Guidance methods are separated into plane and spatial ones. Methods of plane guidance are constructed in the assumption that velocity vectors of the interceptor v and of the target v_t lie in one fixed plane (horizontal, vertical or inclined) at each moment of time during guidance. The methods of spatial guidance do not assume any limitations on the mutual location of velocity vectors of the interceptor and target. In this chapter only methods of plane guidance in a kinematic formulation are examined, that is, without consideration of the forces and moments acting on the aircraft and assuming the presence of complete data on the motion of the target.

Formulation of the Guidance Problem. Let there be a target for which complete information on the motions of its center of mass is given, i.e., the law of motion of the center of mass of the target is given:

$$x_t = f_1(t); \quad z_t = f_2(t)$$

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*Numbers in the margin indicate pagination of the foreign text.

during examination of guidance in a certain fixed Cartesian coordinate system Oxz or the law of change of the angular rate of rotation of the target velocity vector:

$$\omega_t = f(t)$$

during examination of guidance in polar coordinate system TDq. The interceptor, whose law of motion is determined by the selected guidance method, describes a trajectory which depends on the trajectory of the target.

The trajectory characteristics of the interceptor which moves according to /6 a certain guidance method are its curvature 1/r or required g-force n_r, angular velocity of the line "interceptor-target" ω_γ, the target bearing ϕ and target heading q, the distance between the interceptor and target D (see Table 1.1), which determine the feasibility of a specific guidance method both in the theoretical and physical implementation in the guidance system.

The motion of the interceptor and target is analyzed in two coordinate systems: fixed Cartesian Oxz, whose origin is connected to an arbitrary fixed point on the Earth's surface, and relative polar TDq with pole T at the target's center of mass.

Motion equations both for the interceptor and target are written in these coordinate systems. Equations of motion are defined as differential kinematic relations which characterize the change in time of Cartesian or polar coordinates of the aircraft.

In deriving the equations of motion and equations (laws) of guidance methods the following assumptions are made.

1. The aircraft is viewed as a physical point with constant mass.

2. Guidance is performed in a horizontal plane.

3. At any moment (including the initial) the interceptor is in a trajectory which strictly corresponds to the method being examined, i.e., the dynamics of motion of the interceptor on the real trajectory are not taken into consideration (errors of deviation from the trajectory of the method are zero).

Differential equations which describe the motions of the interceptor and target, equation for the guidance method together with the given laws of change of velocities of the interceptor v and the target v_t, target course angle ψ_t (in the Cartesian coordinate system) or the angular velocity of the target ω_γ (in the polar coordinate system), and also with the initial Cartesian coordinates

of the interceptor x_0, z_0 and of the target x_{t0}, z_{t0} or with the polar coordinates of the interceptor D_0, q_0, form a closed equation system.

It is useful to select some natural parameter (for example, the course angle ψ or the target course angle q) and not the time t as the independent variable for integration of this system. Therefore the solutions presented in this chapter are in the form of functions of the indicated parameters. In this case in order to determine the coordinates of the interceptor which flies according to some guidance method (for example, according to the method of triangulation) at moment of time t*, we have to find the course angle ψ^* for this moment of time from the corresponding formulae, and then coordinates from the known course angle ψ^*. One should proceed in similar fashion in determining other parameters.

All methods are described according to a standard system, i.e., determination / of the method is made, a graphic representation of its trajectory is presented, equations of motion and the equation of the guidance method are given, and then solutions for these equations in the two coordinate systems (Cartesian and polar) are given with specific assumptions of the laws of target motion and change of interceptor velocity $v(t)$. The summary of solutions includes: laws of change of Cartesian and polar coordinates of the interceptor and target as functions of course angle ψ or target course angle q, laws of change of range, flight time on a kinematic trajectory, angular velocity of the "interceptor--target" line, radius of curvature r and the kinematic g-force required for flight n_r, the values of these parameters at the point of intercept, and also characteristics of the zone of possible attacks with respect to g-force and velocity.

In this chapter the following guidance methods are given:

1. Pursuit curve.
2. Parallel closing.
3. Constant bearing guidance.
4. Triangulation.
5. Proportional closing.
6. Direct guidance.

1.1. Some Definitions of Trajectory Elements for Guidance

Table 1.1 contains definitions of all trajectory elements encountered in the chapter. These values are indicated in Figures 1.2, 1.6, 1.14, 1.16, 1.25, 1.26.

TABLE 1.1

Element	Definition
Kinematic guidance trajectory	Trajectory determined by the kinematic relations during flight by the guidance method
"Interceptor-target" line (line of sight)	Line segment constructed through interceptor and target points
Center of guidance O (for triangulation)	Origin of the beam which traces the target (point on the line of sight relative to which the method of guidance is determined)
Rate of closing of interceptor and target $\dot{D}$	Rate of change in the distance between the interceptor and target /8
Relative velocity vector of the interceptor (closing velocity vector of interceptor and target $\dot{\mathbf{D}}$	Vector equal to the geometric difference of the velocity vectors of the interceptor v and the target v_t
Relative range vector $\mathbf{D}$	Vector aimed on the "interceptor-target" line at the interceptor and equal in modulus to the distance between the centers of mass of the target and interceptor
Target radius-vector $\mathbf{D}_t$	Vector aimed from the origin of fixed Cartesian coordinate system Oxz at the center of mass of the target and equal in modulus to the distance between the coordinate origin and the center of mass of the target

TABLE 1.1 (CONTINUED)

Element	Definition
Interceptor radius-vector D_i	Vector aimed from the origin of fixed Cartesian coordinate system Oxz at the center of mass of the interceptor and equal in modulus to the distance between the coordinate origin and the center of mass of the interceptor
Guidance Plane	The plane which passes through the velocity vectors of the interceptor and target (if they are coplanar) and in which guidance is performed
Target heading q	Angle between the relative range vector (the "interceptor-target" line) and the target velocity vector
Bearing ϕ	Angle between the relative range vector (the "interceptor-target" line) and the interceptor velocity vector
Interceptor course ψ (target course ψ_t)	The angle between the Ox axis of the Cartesian coordinate system and the velocity vector of the interceptor (target)
Intercept boundary R_i	Distance to the interceptor at the moment of execution of guidance from the origin of the Cartesian coordinate system /9
Time of intercept t_i	Flight time of the interceptor along the kinematic trajectory from the beginning of motion to the moment of execution of guidance
Required g-force n_r	The g-force produced by curvature of the kinematic trajectory and which is required by the interceptor for flight by the guidance method
Zone of possible attacks	The part of the plane around the target in which the interceptor carries out the guidance assignment, while beginning this movement along the kinematic trajectory

TABLE 1.1 (CONTINUED)

Element	Definition
Zone of possible attacks with respect to g-force	The part of the plane around the target from which kinematic trajectories of the interceptor begin which are characterized by the condition $n_r \leq n_{kn}$
Zone of possible attacks with respect to velocity	The part of the plane around the target from which the interceptor begins movement with flight at a given velocity along the kinematic trajectory to encounter the target
Angular velocity of the "interceptor-target" line (line of sight) ω_l	Angular velocity of the relative vector D ("interceptor-target" line) relative to the fixed Cartesian coordinate system Oxz. The positive direction corresponds to clockwise rotation of the end of vector D (see Figures 1.2, 1.6, 1.14, 1.16, 1.25, 1.26)
Angular velocity of the target velocity vector ω_t	Angular velocity ω_t of target velocity vector v_t relative to the fixed Cartesian coordinate system Oxz. The positive direction corresponds to clockwise rotation of the end of vector v_t (see Figures 1.2, 1.6, 1.14, 1.16, 1.25)
Assigned velocity of interceptor velocity vector ω_{kn}	Angular velocity ω_{kn} of the interceptor velocity vector v relative to the fixed Cartesian coordinate system Oxz for a given load value n_{kn}. Positive direction corresponds to clockwise rotation of the end of vector v (see Figures 1.2, 1.6, 1.14, 1.16, 1.25) /10

1.2. Pursuit Curve Method[2]

The method of closing of interceptor on target in which the direction of the tangent to the trajectory of the interceptor at each moment in time passes through the point where the target is located is called the *pursuit curve method.* Equations for the method are given in Table 1.2.

TABLE 1.2

	Equations of Motion	Equations of Guidance Method
Vectorial form	$\mathbf{v}_{rel} = \mathbf{v} - \mathbf{v}_t - \boldsymbol{\omega}_t \times \mathbf{D}$	$\mathbf{D} \times \mathbf{v} = 0$
Cartesian coordinate system	$\dot{x}_t = v_t \cos \psi_t$ $\dot{z}_t = v_t \sin \psi_t$ $\dot{x} = v \cos \psi$ $\dot{z} = v \sin \psi$	$\frac{x - x_t}{z - z_t} = \frac{\dot{x}}{\dot{z}}$
Polar coordinate system	$\dot{D} = -v - v_t \cos q$ $\dot{q} = \frac{v_t \sin q}{D} + \omega_t$	$\varphi(t) \equiv 0$

The kinematic trajectory of the interceptor which corresponds to flight according to the pursuit curve method is called the *pursuit curve.*

1.2.1. Plotting the Kinematic Trajectory by the Pursuit Curve Method /11

Let the target move uniformly with velocity v_t. Its trajectory is divided into segments $T_0T_1 = T_1T_2 = \ldots = v_t$, where Δt is the selected time interval (Figure 1.1). At the initial moment in time the distance from the target to the interceptor D_0 and the target heading q_0 are given. On line I_0T_0, segment I_0I_1, equal in magnitude to $v\Delta t$, is constructed from point I_0. Points I_1 and T_1 are joined by segment I_1T_1. On line I_1T_1 from point I_1 segment I_1I_2 is

[2] Other names for this method are also encountered: optical pursuit curve, "dog curve," "pure pursuit."

again constructed which is equal in value to $v\Delta t$. This construction is then repeated in a similar fashion. The broken line which passes through points I_0, I_1, I_2, ..., makes up the pursuit curve. The smaller the time interval Δt, the closer the constructed line will be to the pursuit curve. As $\Delta t \to 0$, the broken line $I_0 I_1 I_2 I_3$ ... becomes the pursuit curve.

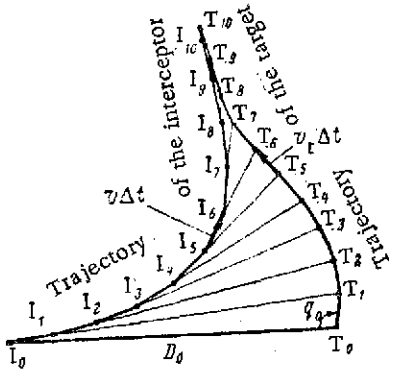

Figure 1.1. Plotting of Kinematic Trajectory According to the Pursuit Curve Method.

1.2.2. Equations of Motion and Pursuit Guidance Method

Let the target move at velocity v_t and turn with an angular velocity ω_t, whose positive direction is indicated in Figure 1.2. At some (initial) moment of time the interceptor is located at point I, and the target is located at point T.

The position of the interceptor and target in the Cartesian coordinate system Oxz is determined by coordinates (x, z) and (x_t, z_t) respectively, and in the polar system TDq by the distance D between the target and interceptor and the target heading q. The orientation of velocity vectors of the interceptor and target in the Cartesian system is determined by the angles ψ and ψ_t between the Ox axis and the corresponding vectors v and v_t.

The equations of motion for the interceptor and target and flight conditions based on the pursuit curve method (equations of the guidance method), which are written in vectorial form and in projections on the axes of the Cartesian and polar systems, are given in Table 1.2.

Initial conditions. For t = 0 coordinates are given: /12

– in the Cartesian system

$$x_t(0) = x_{t0}; \qquad x(0) = x_0;$$

$$z_t(0) = z_{t0}; \qquad z(0) = z_0;$$

– in the polar system

$$D(0) = D_0; \quad q(0) = q_0.$$

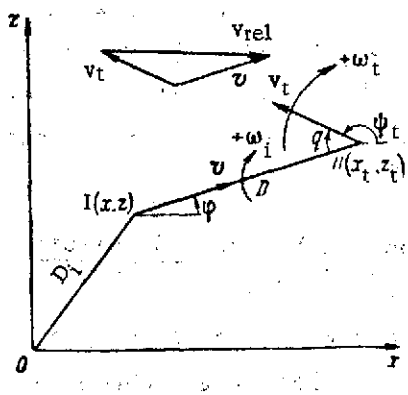

Figure 1.2. Elements of the Kinematic Trajectory of the Pursuit Curve Method.

At this same moment in time angles ψ_0 and ϕ_0 are determined by the relations:

$$\tan\psi_0=\frac{z_{t0}-z_0}{x_{t0}-x_0};\ \varphi_0=0.$$

After we have added to these conditions the laws for change of target velocity speed $v_t(t)$, interceptor velocity speed $v(t)$, target course $\psi_t(t)$ or the angular rate of rotation of target velocity vector $\omega_t(t)$, we obtain a closed system of differential equations both Cartesian and polar.

1.2.3. Solution of Motion Equations for Pursuit Curve Interception by an Interceptor with Constant Velocity (v = const) of Target Flying in Straight Line at Uniform Velocity (ω_t = 0, v_t = const)

Let the target fly rectilinearly and uniformly (ω_t = 0, ψ_t = const, v_t = const). The interceptor solves the guidance problem by flying at constant velocity (v = const). For these conditions the solutions of equations of motion in the polar and Cartesian systems are presented below, as well as the characteristics of the kinematic trajectory of the interceptor and zone of possible attacks.

1. Solution of equations of motion in the polar coordinate system is given by the following formulae. /14

The distance between the interceptor and the target is

$$D=D_0\sin q_0\operatorname{cosec} q\left(\tan\frac{q_0}{2}\cot\frac{q}{2}\right)^k. \qquad (1.1)$$

Values of the functions $(\tan\frac{q}{2})^{\pm k}$ are given in Tables 1.3 and 1.4.

Flight time for the interceptor along the kinematic trajectory from point (D_0, q_0) to point (D, q) is

$$t=\frac{D_0(k-\cos q_0)-D(k-\cos q)}{v_t(k^2-1)}, \qquad (1.2)$$

TABLE 1.3

q in degrees	Values of Function $(\tan \frac{q}{2})^k$															
	k=0,2	k=0,4	k=0,6	k=0,8	k=0,9	k=1,0	k=1,1	k=1,2	k=1,4	k=1,6	k=1,8	k=2,0	k=2,5	k=3,0	k=4,0	k=5,0
10	0,6143	0,3774	0,2318	0,1424	0,1116	0,0875	0,0686	0,0537	0,0330	0,0203	0,0125	0,0076	0,0023	0,00067	0,00007	0,00000
20	0,7069	0,4994	0,3530	0,2495	0,2098	0,1763	0,1483	0,1246	0,0881	0,0622	0,0440	0,0311	0,0130	0,0055	0,0010	0,0002
30	0,7684	0,5905	0,4538	0,3487	0,3057	0,2680	0,2349	0,2058	0,1584	0,1216	0,0935	0,0718	0,0372	0,0192	0,0052	0,0014
40	0,8170	0,6674	0,5454	0,4455	0,4027	0,3640	0,3290	0,2974	0,2452	0,1985	0,1622	0,1325	0,0800	0,0516	0,0176	0,0064
50	0,8584	0,7365	0,6313	0,5432	0,5031	0,4663	0,4320	0,4002	0,3436	0,2950	0,2532	0,2174	0,1484	0,1014	0,0472	0,0220
60	0,8960	0,8028	0,7191	0,6443	0,6099	0,5773	0,5464	0,5184	0,4634	0,4152	0,3720	0,3332	0,2532	0,1924	0,1110	0,0641
70	0,9313	0,8672	0,8076	0,7485	0,7258	0,7003	0,6758	0,6522	0,6073	0,5655	0,5266	0,4905	0,4104	0,3435	0,2405	0,1685
80	0,9657	0,9324	0,9001	0,8708	0,8541	0,8393	0,8247	0,8104	0,7825	0,7554	0,7295	0,7044	0,6454	0,5912	0,4962	0,4164
90	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000

Note: Commas indicate decimal points.

TABLE 1.4

q in degrees	Values of Function $(\tan \frac{q}{2})^{-k}$															
	k=0,2	k=0,4	k=0,6	k=0,8	k=0,9	k=1,0	k=1,1	k=1,2	k=1,4	k=1,6	k=1,8	k=2,0	k=2,5	k=3,0	k=4,0	k=5,0
10	1,628	2,650	4,313	7,021	8,958	11,43	14,58	18,61	30,28	49,29	80,24	102,4	130,6	4416	14930	—
20	1,415	2,002	2,8324	4,009	4,767	5,671	6,746	8,024	11,36	16,07	22,74	32,17	76,6	182,4	1035	5868
30	1,301	1,693	2,2035	2,868	3,271	3,372	4,257	4,856	6,320	8,222	10,70	13,92	26,91	51,96	194	724
40	1,095	1,1996	1,314	1,440	1,507	1,578	1,651	1,728	1,892	2,073	2,271	2,487	3,125	3,923	6,189	9,76
50	1,165	1,357	1,580	1,841	1,987	2,145	2,314	2,498	2,911	3,390	3,949	4,601	6,738	9,865	21,16	45,39
60	1,116	1,246	1,391	1,552	1,640	1,732	1,830	1,933	2,158	2,403	2,689	3,000	3,950	5,198	9,024	15,60
70	1,0738	1,153	1,238	1,329	1,378	1,428	1,480	1,533	1,646	1,768	1,899	2,039	2,437	2,912	4,157	5,94
80	1,0356	1,073	1,111	1,151	1,170	1,191	1,212	1,234	1,278	1,323	1,371	1,420	1,550	1,691	2,016	2,40
90	1,0000	1,0000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,00

Note: Commas indicate decimal points.

or

$$t=\frac{D_0}{v_t}\frac{k-\cos q_0}{k^2-1}\left[1-\frac{k-\cos q}{k-\cos q_0}\sin q_0\operatorname{cosec} q\left(\tan\frac{q_0}{2}\cdot\cot\frac{q}{2}\right)^k\right]. \tag{1.2'}$$

The angular velocity of the "interceptor-target" line is

$$\omega_l=\dot{q}=\frac{v_t}{D_0}\sin^2 q\operatorname{cosec} q_0\left(\cot\frac{q_0}{2}\cdot\tan\frac{q}{2}\right)^k. \tag{1.3}$$

The solution for *special cases* of the target heading values.

a) The target heading q is quite small ($\sin q \approx q$; $\cos q \approx 1$). The solution is given by the following expressions.

The distance between the interceptor and target is

$$D=D_0-v_t(k+1)t. \tag{1.4}$$

The target heading is

$$q=q_0\left[1+\frac{v_t(k+1)t}{D_0}\right]^{\frac{1}{k}}. \tag{1.5}$$

b) The target heading q is quite close to $\pi/2$ ($\sin q \approx 1$; $\cos q \approx q$). The solution is given by the following expressions.

The distance between the interceptor and target is

$$D=D_0\exp\left[-k(q-q_0)-\frac{1}{2}(q^2-q_0^2)\right]. \tag{1.6}$$

Flight time of the interceptor along the kinematic trajectory from (D_0, q_0) to point (D, q) is

$$t=\sqrt{\frac{\pi}{2}}\frac{D}{v_{t\vartheta}}[\Phi(k+q)-\Phi(k+q_0)]\exp\frac{(k+q_0)^2}{2}, \tag{1.7}$$

where $\Phi(k+q)=\sqrt{\frac{2}{\pi}}\int_0^{k+q} e^{-\frac{x^2}{2}}dx$ is the probability integral.

2. Solution of equations of motion in the Cartesian coordinate system is given by the following formulae. /15

The coordinates of the interceptor are

$$\left.\begin{aligned}
x=x_0+\frac{v\cos\psi_{t0}}{\dot\psi_0(k^2-1)}\,\frac{\sin^2(\psi_0-\psi_{t0})}{\sin(\psi-\psi_{t0})}[1+k\cos(\psi-\psi_{t0})]\times\\
\times\left(\cot\frac{\psi_0-\psi_{t0}}{2}\tan\frac{\psi-\psi_{t0}}{2}\right)^k\left[1-\frac{1+k\cos(\psi_0-\psi_{t0})}{1+k\cos(\psi-\psi_{t0})}\times\right.\\
\left.\times\frac{\sin(\psi-\psi_{t0})}{\sin(\psi_0-\psi_{t0})}\left(\cot\frac{\psi_0-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k\right]-\\
-\frac{v_t\sin\psi_{t0}}{\dot\psi_0}\sin^2(\psi_0-\psi_{t0})\left[1-\left(\cot\frac{\psi-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k\right]\times\\
\times\left(\cot\frac{\psi_0-\psi_{t0}}{2}\tan\frac{\psi-\psi_{t0}}{2}\right)^k;\\
z=z_0+\frac{v\sin\psi_{t0}}{\dot\psi_0(k^2-1)}\,\frac{\sin^2(\psi_0-\psi_{t0})}{\sin(\psi-\psi_{t0})}[1+k\cos(\psi-\psi_{t0})]\times\\
\times\left(\cot\frac{\psi_0-\psi_{t0}}{2}\tan\frac{\psi-\psi_{t0}}{2}\right)^k\left[1-\frac{1+k\cos(\psi_0-\psi_{t0})}{1+k\cos(\psi-\psi_{t0})}\times\right.\\
\left.\times\frac{\sin(\psi-\psi_{t0})}{\sin(\psi_0-\psi_{t0})}\left(\cot\frac{\psi-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k\right]+\\
+\frac{v_t\cos\psi_{t0}}{\dot\psi_0}\sin^2(\psi_0-\psi_{t0})\left[1-\left(\cot\frac{\psi-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k\right]\times\\
\times\left(\cot\frac{\psi_0-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k.
\end{aligned}\right\}\qquad(1.8)$$

Coordinates of the target are

$$\left.\begin{aligned}
x_t=x_{t0}+v_t t\cos\psi_{t0};\\
z_t=z_{t0}+v_t t\sin\psi_{t0}.
\end{aligned}\right\}\qquad(1.9)$$

Flight time for the interceptor along the kinematic trajectory from (x_0, z_0) to point (x, z) is

$$\begin{aligned}
t=\frac{1}{\dot\psi_0}\left(\cot\frac{\psi-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k[k+\cos(\psi-\psi_{t0})]\sin^2(\psi-\psi_{t0})\times\\
\times\operatorname{cosec}^2(\psi_t-\psi_{t0})\left[1-\frac{k+\cos(\psi_0-\psi_{t0})}{k+\cos(\psi-\psi_{t0})}\,\frac{\sin(\psi-\psi_{t0})}{\sin(\psi_0-\psi_{t0})}\times\right.\\
\left.\times\left(\cot\frac{\psi-\psi_{t0}}{2}\tan\frac{\psi_0-\psi_{t0}}{2}\right)^k\right].
\end{aligned}\qquad(1.10)$$

The angular rate of rotation of the "interceptor-target" line is

$$\omega_l = \dot{\psi} = \dot{\psi}_0 \left(\cot \frac{\psi - \psi_{t0}}{2} \tan \frac{\psi_0 - \psi_{t0}}{2} \right)^k \sin^2(\psi - \psi_{t0}) \operatorname{cosec}^2(\psi_0 - \psi_{t0}). \tag{1.11}$$

The square of the distance of the interceptor from the origin of the Cartesian coordinate system is /16

$$\begin{aligned}
D_1^2 = {} & x_0^2 + z_0^2 + \frac{k^2 D_0^2}{(k^2-1)^2} \left(\cot \frac{\psi_0 - \psi_{t0}}{2} \tan \frac{\psi - \psi_{t0}}{2} \right)^{2k} \times \\
& \times \sin^2(\psi_0 - \psi_{t0}) \operatorname{cosec}^2(\psi - \psi_{t0}) [1 + k \cos(\psi - \psi_{t0})]^2 \times \\
& \times \left[1 - \frac{1 + k \cos(\psi_0 - \psi_{t0})}{1 + k \cos(\psi - \psi_{t0})} \frac{\sin(\psi - \psi_{t0})}{\sin(\psi_0 - \psi_{t0})} \cot \frac{\psi - \psi_{t0}}{2} \tan \frac{\psi_0 - \psi_{t0}}{2} \right]^2 + \\
& + D_0^2 \left(\cot \frac{\psi_0 - \psi_{t0}}{2} \tan \frac{\psi - \psi_{t0}}{2} \right)^2 \left(1 - \cot \frac{\psi - \psi_{t0}}{2} \tan \frac{\psi_0 - \psi_{t0}}{2} \right]^2 \times \\
& \times \sin(\psi_0 - \psi_{t0}) - \frac{2kD_0}{k^2-1} \frac{\sin(\psi_0 - \psi_{t0})}{\sin(\psi - \psi_{t0})} \cot \frac{\psi_0 - \psi_{t0}}{2} \tan \frac{\psi - \psi_{t0}}{2} \times \\
& \times [1 + k \cos(\psi - \psi_{t0})] (x_0 \cos \psi_{t0} + z_0 \sin \psi_{t0}) \times \\
& \times \left[1 - \frac{1 + k \cos(\psi_0 - \psi_{t0})}{1 + k \cos(\psi - \psi_{t0})} \frac{\sin(\psi - \psi_{t0})}{\sin(\psi_0 - \psi_{t0})} \cot \frac{\psi - \psi_{t0}}{2} \tan \frac{\psi_0 - \psi_{t0}}{2} \right] + \\
& + 2D_0 \cot \frac{\psi_0 - \psi_{t0}}{2} \tan \frac{\psi - \psi_{t0}}{2} \left(1 - \cot \frac{\psi - \psi_{t0}}{2} \tan \frac{\psi_0 - \psi_{t0}}{2} \right) \times \\
& \times (x_0 \sin \psi_{t0} - z_0 \cos \psi_{t0}) \sin(\psi_0 - \psi_{t0}).
\end{aligned} \tag{1.12}$$

The solution for *special cases* of target motion.

a) The target moves parallel to the axis Ox ($\psi_{t0} = 0$). The solution is given by the following expressions.

Coordinates of the interceptor are

$$\left.\begin{aligned}
x = {} & x_0 + \frac{v \sin^2 \psi_0}{\dot{\psi}_0 (k^2-1) \sin \psi} (1 + k \cos \psi) \left(\cot \frac{\psi_0}{2} \tan \frac{\psi}{2} \right)^k \times \\
& \times \left[1 - \frac{1 + k \cos \psi_0}{1 + k \cos \psi} \sin \psi \operatorname{cosec} \psi_0 \left(\cot \frac{\psi}{2} \tan \frac{\psi_0}{2} \right)^k \right]; \\
z = {} & z_0 + \frac{v \sin^2 \psi_0}{\dot{\psi}_0} \left(\cot \frac{\psi_0}{2} \tan \frac{\psi}{2} \right)^k \left[1 - \cot \frac{\psi}{2} \tan \frac{\psi_0}{2} \right)^k \Big].
\end{aligned}\right\} \tag{1.13}$$

Flight time of the interceptor along the kinematic trajectory from point (x_0, z_0) to point (x, z) is

$$t=\frac{\sin^2\psi_0}{\dot{\psi}_0(k^2-1)}\left(\cot\frac{\psi_0}{2}\right)^k\left(\tan\frac{\psi}{2}\right)^{k-1}\times$$
$$\times\left[1-\frac{k+\cos\psi_0}{k+\cos\psi}\sin\psi\operatorname{cosec}\psi_0\left(\cot\frac{\psi}{2}\tan\frac{\psi_0}{2}\right)^k\right]. \tag{1.14}$$

The angular velocity for the "interceptor-target" line is

$$\omega_\zeta=\dot{\psi}=\dot{\psi}_0\sin^2\psi\operatorname{cosec}^2\psi_0\left(\cot\frac{\psi}{2}\tan\frac{\psi_0}{2}\right)^k. \tag{1.15}$$

b) The target moves parallel to the axis Oz ($\psi_{t0}=\pi/2$). The solution is given by the following expressions. /17

Coordinates of the interceptor are

$$\left.\begin{aligned}x&=x_0-\frac{v\sin^2\psi_0}{\dot{\psi}_0(k^2-1)}\operatorname{cosec}\psi(1+k\cos\psi)\left(\cot\frac{\psi_0}{2}\tan\frac{\psi}{2}\right)^k\times\\&\times\left[1-\frac{1+k\cos\psi_0}{1+k\cos\psi}\sin\psi\operatorname{cosec}\psi_0\left(\cot\frac{\psi}{2}\tan\frac{\psi_0}{2}\right)^k\right];\\z&=z_0-\frac{v}{\dot{\psi}_0}\sin^2\psi_0\left(\cot\frac{\psi_0}{2}\tan\frac{\psi}{2}\right)^k\left[1-\left(\cot\frac{\psi}{2}\tan\frac{\psi_0}{2}\right)^k\right].\end{aligned}\right\} \tag{1.16}$$

Flight time for the interceptor along the kinematic trajectory from point (x_0, z_0) to point (x, z)

$$t=-\frac{\cos^2\psi_0}{\dot{\psi}_0(k^2-1)}\cot^k\left(\frac{\pi}{4}-\frac{\psi_0}{2}\right)\tan^k\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\times$$
$$\times\left[1-\frac{k+\sin\psi_0}{k+\sin\psi}\cos\psi\sec\psi_0\cot^k\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\tan^k\left(\frac{\pi}{4}-\frac{\psi_0}{2}\right)\right]. \tag{1.17}$$

The angular velocity of the "interceptor-target" line is

$$\omega_\zeta=\dot{\psi}_0\cos^2\psi\sec^2\psi_0\cot^k\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\tan^k\left(\frac{\pi}{4}-\frac{\psi_0}{2}\right). \tag{1.18}$$

3. The relation between the elements of the kinematic trajectory of the interceptor in the polar and Cartesian systems is:

$$\omega_1 = q = \dot{\psi} = -\frac{v_t}{D}\sin(\psi - \psi_{t0}) = \frac{v_t}{D}\sin q; \tag{1.19}$$

$$D = -v_t[k - \cos(\psi - \psi_{t0})] = -v_t(k + \cos q); \tag{1.20}$$

$$q = \pi + \psi - \psi_{t0}. \tag{1.21}$$

4. Radius of curvature of the kinematic trajectory of the interceptor r and the required g-force for the interceptor n_r for flight on the pursuit curve is determined by formulae

$$r = \frac{kD}{\sin q} = \frac{kD_0}{\sin q_0}\cot^k\frac{q}{2}\tan^k\frac{q_0}{2}\sin^2 q_0\operatorname{cosec}^2 q =$$
$$= \frac{v}{\dot{q}} = \frac{v}{\dot{\psi}}\cot^k\frac{\psi_0 - \psi_{t0}}{2}\tan^k\frac{\psi - \psi_{t0}}{2}\sin^2(\psi_0 - \psi_{t0})\operatorname{cosec}^2(\psi - \psi_{t0}); \tag{1.22}$$

$$n_r = \frac{vv_t\sin q}{g_0 D} = \frac{vv_t}{g_0 D_0}\cot^k\frac{q_0}{2}\tan^k\frac{q}{2}\cdot\sin q\operatorname{cosec} q_0. \tag{1.23}$$

5. Parameters of motion at the moment of intercept (D = 0, x = x_t, z = z_t) are represented by the following expressions.

Rate of closing is /18

$$D_k = -v - v_t.$$

Time of intercept is

$$t_i = \frac{D_0(k - \cos q_0)}{v_t(k^2 - 1)} = \frac{D_0}{v_t}\cdot\frac{k + \cos(\psi_0 - \psi_{t0})}{k^2 - 1}. \tag{1.25}$$

Square of the limit of intercept is

$$R_i^2 = x_{t0}^2 + z_{t0}^2 + D_0^2\left[\frac{k + \cos(\psi - \psi_{t0})}{k^2 - 1}\right]^2 +$$
$$+ 2D_0\frac{k + \cos(\psi - \psi_{t0})}{k^2 - 1}(x_{t0}\cos\psi_{t0} + z_{t0}\sin\psi_{t0}) = \tag{1.26}$$

$$= x_0^2 + z_0^2 + D_0^2 \sin^2(\psi_0 - \psi_{t0}) + D_0^2 \frac{k^2}{(k^2-1)^2}[1 + k\cos(\psi_0 - \psi_{t0})]^2 +$$
$$+ 2D_0 \frac{k}{k^2-1}[1 + k\cos(\psi_0 - \psi_{t0})](x_0 \cos\psi_{t0} + z_0 \sin\psi_{t0}) -$$
$$- 2D_0 \sin(\psi_0 - \psi_{t0}) \cdot (x_0 \sin\psi_{t0} - z_0 \cos\psi_{t0}). \qquad (1.26)$$

Coordinates of intercept are

$$\left.\begin{aligned} x_i &= x_0 - \frac{v\cos\psi_{t0}}{\dot{\psi}_0(k^2-1)}[1 + k\cos(\psi_0 - \psi_{t0})]\sin(\psi_0 - \psi_{t0}) + \\ &+ \frac{v\sin\psi_{t0}}{k\dot{\psi}_0}\sin^2(\psi_0 - \psi_{t0}); \\ z_i &= z_0 - \frac{v\sin\psi_{t0}}{\dot{\psi}_0(k^2-1)}[1 + k\cos(\psi_0 - \psi_{t0})]\sin(\psi_0 - \psi_{t0}) - \\ &- \frac{v\cos\psi_{t0}}{k\dot{\psi}_0}\sin^2(\psi_0 - \psi_{t0}). \end{aligned}\right\} \qquad (1.27)$$

The angular velocity of the "interceptor-target" line ω_l and the required g-force n_r are determined from Table 1.5.

TABLE 1.5

Ratio of the Velocities of Interceptor and Target k	"Interceptor-Target" Line ω_l	Required g-Force n_r
$1 < k < 2$	$\omega_l = 0$	$n_r = 0$
$k = 2$	$\omega_l = 4\omega_{l0}$	$n_r = 4\omega_{l0}\frac{v}{g_0}$
$k > 2$	$\omega_l = \infty$	$n_r = \infty$

6. Zone of possible attacks with respect to g-force. Boundary of the zone of possible attacks with respect to g-force (see Table 1.1) is determined by the equation /19

$$D_b = \frac{vv_t \sin q}{g_0 n \, kn}.$$

The boundary of the zone of possible attacks is formed by two circles tangent at the point of the target with radii $vv_t/2g_0n_{kn}$. Zone of possible attacks with respect to g-force is shown in Figure 1.3. For calculating the boundary of the zone of possible attacks a graph of the function

$f_1=\frac{\sin q}{g_0 n_{kn}}$ is given in Figure 1.4. Then the boundary is determined by the relation

$$D_b = f_1 v v_t .$$

q, degree	10	30	50	90
f_1, sec^2/m (according to Figure 1.4)	0.0045	0.013	0.020	0.026
D_b, meters	935	2705	4160	5410

Example 1. The target flies at a velocity v_t = 1500 km/hour. The interceptor flies on the pursuit curve at v = 1800 km/hr with g-force n_{kn} = 4. Determine the boundary of the zone of possible attacks with respect to g-force. The function f_1 is found for selected values of target headings q = 10°, 30°, 50°, 90° and n_{kn} = 4 from the graph in Figure 1.4. Then D_b is calculated for selected values of the target heading.

1.2.4. Solution of Equations of Motion for Pursuit Curve Interception by Interceptor Flying at Constant Velocity (v = const) of Target Flying Uniformly in a Logarithmic Spiral ($\psi - \psi_t$ = const, v_t = const)

Let the target move on a curve described in the Cartesian coordinate system by equations

$$\left.\begin{aligned} x_t &= x_{t0}+\frac{v_t}{\dot{\psi}_0}\left[e^{(\psi_t-\psi_{t0})\tan\delta}-1\right]\sin(\psi_t+\delta)\cos\delta;\\ z_t &= z_{t0}+\frac{v_t}{\dot{\psi}_0}\left[e^{(\psi_t-\psi_{t0})\tan\delta}-1\right]\cos(\psi_t+\delta)\cos\delta, \end{aligned}\right\} \qquad (1.28)$$

where

$$\tan\delta = \frac{k - \cos(\psi_0 - \psi_{t0})}{\sin(\psi_0 - \psi_{t0})} = \text{const.} \qquad (1.29)$$

The trajectory of the target is a logarithmic spiral with a radius of curvature /20

$$r_t = \frac{v_t}{\dot{\psi}_0} e^{(\psi_{t0} - \psi_t)\tan\delta} \qquad (1.30)$$

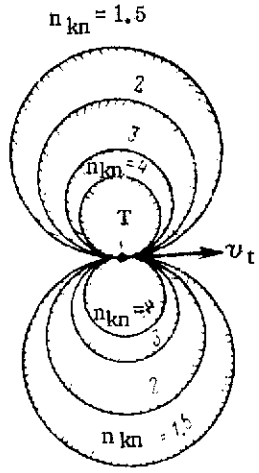

Figure 1.3. The Zone of Possible Attacks With Respect to g-Force for the Pursuit Curve Method During Guidance to a Rectilinearly Flying Target (Zone of Possible Attacks is Located Outside the Crosshatched Area, Bounded by the Circles n_{kn} = const.

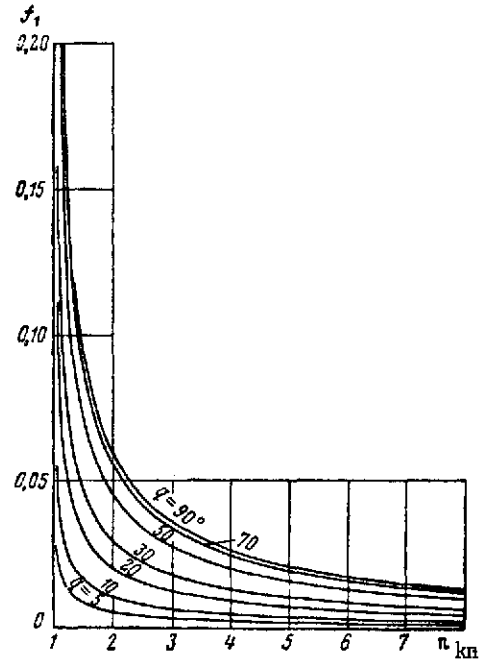

Figure 1.4. Graph of the Function $f_1 = \frac{\sin q}{q_0 n_{kn}}$.

1. Solution of the equations of motion (Table 1.2) in the Cartesian coordinate system is given by the following formulae. /21

Coordinates of the interceptor are

$$\left.\begin{aligned} x &= x_0 + \frac{v}{\dot{\psi}_0}\left[e^{(\psi-\psi_0)\tan\delta} - 1\right]\sin(\psi_0+\delta)\cos\delta; \\ z &= z_0 - \frac{v}{\dot{\psi}_0}\left[e^{(\psi-\psi_0)\tan\delta} - 1\right]\cos(\psi_0+\delta)\cos\delta. \end{aligned}\right\} \qquad (1.31)$$

The trajectory of the interceptor is also a logarithmic spiral.

The flight time of the interceptor on the kinematic trajectory from point (x_0, z_0) to point (x, z) is

$$t = \frac{\cot\delta}{\dot{\psi}_0}\left[e^{(\psi-\psi_0)\tan\delta} - 1\right]. \tag{1.32}$$

The angular velocity of the "interceptor-target" line is

$$\omega_{\zeta} = \dot{\psi}_0 e^{(\psi_{t0}-\psi_t)\tan\delta}. \tag{1.33}$$

2. The connection of elements in the kinematic trajectory of the interceptor at the initial moment of time is

$$\left.\begin{aligned} \psi_0 &= \arctan\frac{z_{t0}-z_0}{x_{t0}-x_0}; \\ \dot{\psi}_0 &= \dot{\psi}_{t0} = \frac{v_t\sin(\psi_{t0}-\psi_0)}{\sqrt{(x_{t0}-x_0)^2+(z_{t0}-z_0)^2}}. \end{aligned}\right\}$$

3. The radius of curvature of the kinematic trajectory r and the required g-force of the interceptor n_r for the flight on the pursuit curve is determined by formulae

$$r = \frac{v}{\dot{\psi}_0} e^{(\psi-\psi_0)\tan\delta}; \tag{1.35}$$

$$n_r = \frac{v\dot{\psi}_0}{g_0} e^{(\psi_0-\psi)\tan\delta}. \tag{1.36}$$

4. The parameters of motion at the moment of intercept are represented by the following expressions.

For $\delta = 0$, i.e., when $k = \cos(\psi_t - \psi) < 1$, the target moves in a circle, but the pursuit curve is also a circle which is concentric with the first one. In this case the trajectory of the interceptor does not intersect the target trajectory.

For $\delta \neq 0$, i.e., when $k > \cos(\psi_t - \psi)$, the parameters of motion are found from the following relations.

The course of the interceptor is

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$$\psi_к = \psi_0 + \cot\delta \cdot \ln\left[1 - \frac{D_0\dot{\psi}_0}{v_t \sin(\psi_{t0} - \psi_0)}\right]. \tag{1.37}$$

The time of interception is

$$t_к = \frac{D_0}{v_t\,[k - \cos(\psi_{t0} - \psi_0)]}. \tag{1.38}$$

The interception limit is

$$R_к = [x_0^2 + z_0^2 + k^2 D_0^2 \cos^2\delta \operatorname{cosec}^2(\psi_{t0} - \psi_0) + \\ + 2kD_0 \cos\delta \operatorname{cosec}(\psi_{t0} - \psi_0)[z_0 \cos(\psi_0 + \delta) - x_0 \sin(\psi_0 + \delta)]]^{\frac{1}{2}}, \tag{1.39}$$

where δ is determined from formula (1.29).

The coordinates of the intercept point are

$$x_к = x_0 - kD_0 \sin(\psi_0 + \delta)\cos\delta \operatorname{coses}(\psi_{t0} - \psi_0); \\ z_к = z_0 + kD_0 \cos(\psi_0 + \delta)\cos\delta \operatorname{cosec}(\psi_{t0} - \psi_0). \tag{1.40}$$

1.3. Parallel Closing Method

The guidance method in which the "interceptor-target" line remains parallel to its initial position (angular velocity of the "interceptor-target" line is zero) is called the *parallel closing method*.

During rectilinear and uniform flight of the target the kinematic trajectory of the interceptor which flies at a constant velocity is a straight line passing through a point where interception should occur.

Equations of this method are given in Table 1.6.

1.3.1. Plotting the Kinematic Trajectory for the Parallel Approach Method

Let us assume that the target moves uniformly at velocity v_t. The distance D_0 from the target to the interceptor and the target heading q_0

at the initial moment of time are given. The trajectory of the target (Figure 1.5) is divided into equal segments $T_0T_1 = T_1T_2 = T_2T_3 = \ldots = v_t\Delta t$, where Δt is the selected interval of time. A straight line is drawn through point T_0 at angle q_0 to segment T_0T_1. Segment $T_0I_0 = D_0$ is marked off on it. Straight lines which are parallel to the initial position of the "interceptor-target" line I_0T_0 are drawn through points T_1, T_2, T_3... . Then point I_1 of intersection of the straight parallel segment I_0T_0 passing through point T_1, and circle of radius $v\Delta t$, with center at point I_0, is determined. Points I_0 and T_1 are connected by a segment. The point I_2 of intersection of the line which passes through point T_2 parallel to segment I_0T_0 and a circle of radius $v\Delta t$ with center at point I_1 is found. Points I_1 and T_1 are connected /23 by a segment. Points I_2, I_3, ... are determined by similar construction. The line which passes through points I_0, I_1, I_2 ..., when $\Delta t \to 0$ is the *parallel closing trajectory.*

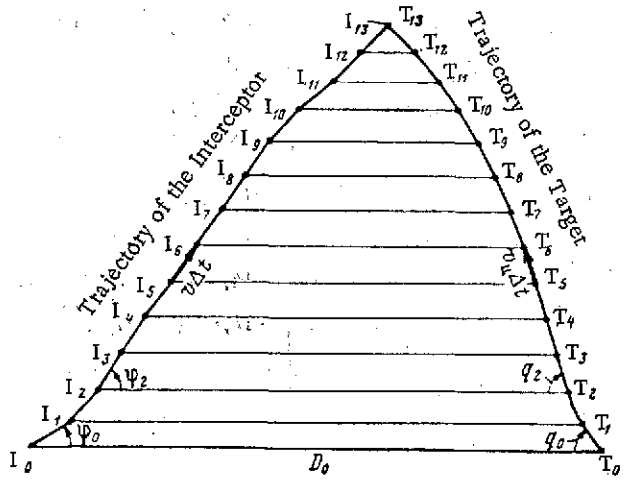

Figure 1.5. Plotting Kinematic Trajectory by the Parallel Closing Method.

1.3.2. Equations of Motion and of Parallel Closing Guidance Method

Assume that the target moves at velocity v_t and turns with angular velocity ω_t, whose positive direction is shown in Figure 1.6.

The positions of the interceptor and target in Cartesian coordinate system Oxz, whose origin is located at arbitrary point O and whose axes Ox and Oz are oriented arbitrarily, are determined by coordinates (x, z) and (x_t, z_t), respectively.

The position of the interceptor relative to the target in polar coordinate system TDq, whose origin is connected with the target, is determined by the distance D between the target and the interceptor and the target heading q, which determines the angular position of the interceptor relative to the target velocity vector.

TABLE 1.6

	Equations of Motion	Equations of the Guidance Method
Vectorial form	$v_{rel} = v - v_t - \omega_t \times D$	$v_{rel} \times D = 0$
Cartesian coordinate system	$\dot{x}_t = v_t \cos \psi_t$ $\dot{z}_t = v_t \sin \psi_t$ $\dot{x} = v \cos \psi$ $\dot{z} = v \sin \psi$	$\frac{x - x_t}{z - z_t} = \frac{x_0 - x_{t0}}{z_0 - z_{t0}}$
Polar coordinate system	$\dot{D} = -v \cos \phi - v_t \cos q$ $\dot{q} = \omega_t$	$\sin \phi = \frac{1}{k} \sin q$

The orientation of velocity vectors of the interceptor v and target v_t in Cartesian coordinate system Oxz is given by angles ψ and ψ_t between the axes Ox and the corresponding vector v or v_t.

The orientation of the velocity vector of interceptor v in the polar system is given by the target bearing angle ϕ between the "interceptor-target" line and the velocity vector v.

The equations of motion for the interceptor and target and the flight conditions of the interceptor by the parallel closing method, written in vectorial form, and also in projections onto the axes of Cartesian and polar systems, are given in Table 1.6. /24

Initial conditions. When t = 0 the following coordinates are given:

— in the Cartesian system

$$x_t(0) = x_{t0}; \qquad x(0) = x_0;$$

$$z_t(0) = z_{t0}; \qquad z(0) = z_0;$$

— in the polar system

$$D(0) = D_0; \qquad q(0) = q_0.$$

At this same moment of time angles ψ_0 and ϕ_0 are determined by the relations

$$\sin(\varepsilon_0 - \psi_0) = \frac{1}{k}\sin(\varepsilon_0 - \phi_{t0});$$

$$\sin\varphi_0 = \frac{1}{k}\sin q_0;$$

$$\tan\varepsilon_0 = \frac{z_{t0} - z_0}{x_{t0} - x_0}.$$

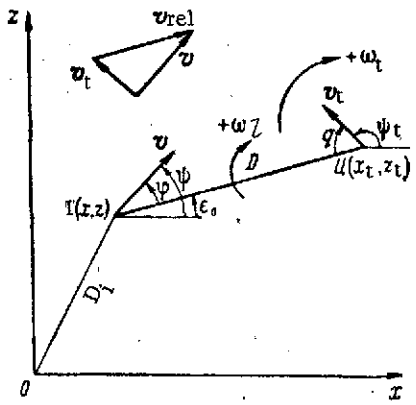

Figure 1.6. Kinematic Trajectory Elements of Parallel Closing Method.

By adding the laws of change in target velocity $v_t(t)$, velocity of the interceptor $v(t)$, target course $\psi_t(t)$ or angular rate of rotation of the target velocity vector $\omega_t(t)$ to these conditions, we obtain a closed system of differential equations in both the Cartesian and polar coordinates.

1.3.3. Solution of Equations of Motion for Interception by Parallel Closing Method /25

Assume that the target flies in a straight line rectilinearly ($\omega_t = 0$, $\psi_t = \text{const}$) at constant velocity ($v_t = \text{const}$), and that the interceptor flies at constant velocity ($v = \text{const}$).

The solution of equations of motion under these conditions for the polar and Cartesian coordinate systems, and for certain characteristics of the kinematic trajectory and zone of attack are given below.

1. The solution of equations of motion in the polar coordinate system is given by the following formulae.

The distance between the interceptor and target

$$D = D_0 - v_t t\,(\cos q_0 - k\cos\phi_0). \qquad (1.41)$$

The flight time of the interceptor on the kinematic trajectory from point (D_0, q_0) to point (D, q) is

$$t = \frac{D_0 - D}{v_t (\cos q_0 - k \cos \varphi_0)} . \tag{1.42}$$

Target bearing is

$$\varphi = \arcsin \left(\frac{1}{k} \sin q_0 \right) . \tag{1.43}$$

The target course is

$$q = q_0 . \tag{1.44}$$

2. The solution of equations of motion in the Cartesian coordinate system is given by the following formulae.

Coordinates of the interceptor

$$\left.\begin{aligned} x &= x_0 + vt \cos \psi_0; \\ z &= z_0 + vt \sin \psi_0. \end{aligned}\right\} \tag{1.45}$$

Coordinates of the target

$$\left.\begin{aligned} x_t &= x_{t0} + v_t t \cos \psi_{t0}; \\ z_t &= z_{t0} + v_t t \sin \psi_{t0}. \end{aligned}\right\} \tag{1.46}$$

Flight time of the interceptor on the kinematic trajectory from point (x_0, z_0) to point (x, z) is

$$t = \frac{x - x_0}{v \cos \psi_0} = \frac{z - z_0}{v \sin \psi_0} . \tag{1.47}$$

The distance of the interceptor from the origin of the coordinates is

$$D_i = \sqrt{x_0^2 + z_0^2 + v^2 t^2 + 2vt (x_0 \cos \psi_0 + z_0 \sin \psi_0)} . \tag{1.48}$$

The equations for the trajectories of the interceptor and target are /26

$$z = z_0 = x_0 \tan \psi_0 + x \tan \psi_0;$$

$$z_t = z_{t0} = x_{t0} \tan \psi_{t0} + x_t \tan \psi_{t0}, \tag{1.49}$$

where

$$\psi_0 = \varepsilon_0 - \arcsin \left[\frac{1}{k} \sin (\varepsilon_0 - \psi_{t0})\right];$$

$$\varepsilon_0 = \arctan \frac{z_0 - z_{t0}}{x_0 - x_{t0}}. \tag{1.50}$$

3. Parameters of motion at the moment of interception ($D = 0$, $x = x_t$, $z = z_t$) are given by the following expressions.

The time of interception is

$$t_k = \frac{D_0}{v_t (\cos q_0 + k \cos \phi_0)} = \frac{x_0 - x_{t0}}{v_t (\cos \psi_{t0} - k \cos \psi_0)} = \frac{z_0 - z_{t0}}{v_t (\sin \psi_{t0} - k \sin \psi_0)}. \tag{1.51}$$

The boundary of interception is

$$R_k = [(x_0 \cos \psi_{t0} - kx_{t0} \cos \psi_0)^2 (z_0 - z_{t0})^2 + (z_0 \sin \psi_{t0} - kz_{t0} \sin \psi_0)^2 \times$$
$$\times (x_0 - x_{t0})^2]^{1/2} [(z_0 - z_{t0}) (\cos \psi_{t0} - k \cos \psi_0)]^{-1}. \tag{1.52}$$

Coordinates of the point of interception are

$$\left.\begin{aligned} x_к &= \frac{x_0 \cos \psi_{t0} - k x_{t0} \cos \psi_0}{\cos \psi_{t0} - k \cos \psi_0}; \\ z_к &= \frac{z_0 \sin \psi_{t0} - k x_{t0} \sin \psi_0}{\sin \psi_{t0} - k \sin \psi_0}. \end{aligned}\right\} \tag{1.53}$$

4. The zone of attack with respect to velocity. When interceptor velocity is greater than target velocity ($k > 1$), the zone of attack with respect to velocity (see Table 1.1) is the entire guidance plane. When the velocity of the interceptor is less than the target velocity ($k \leqslant 1$), the zone of attack with respect to velocity is determined by the inequality in the range of target course angles $-\pi/2 \leqslant q \leqslant \pi/q$:

$$q \leqslant \arcsin k. \tag{1.54}$$

For target courses $\pi \geqslant q \geqslant \pi/2$ and $-\pi/2 \geqslant q \geqslant -\pi$ there is no zone of attack.

Figure 1.7 gives the zone of attack with respect to velocity. This zone is valid for any distance between the interceptor and target for a given ratio of interceptor and target velocities. /27

Figure 1.8 shows the range of values for the ratio of interceptor and target velocities k, at which the interceptor meets the target, depending on the target heading q at the beginning of interceptor motion.

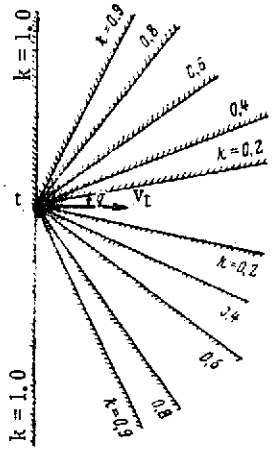

Figure 1.7. The Zone of Attack with Respect to Velocity for the Parallel Closing Method During Guidance of an Interceptor to a Rectilinearly Moving Target (The Zone is Located Outside the Hatched Area, Bounded by Straight Lines k = = const).

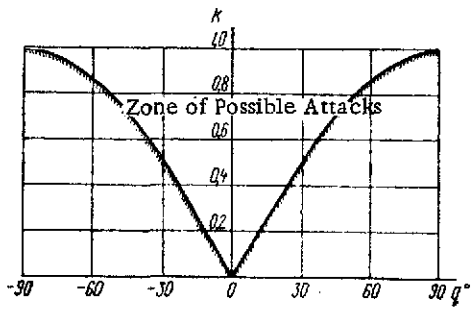

Figure 1.8. Range of Velocity Ratios k and Target Heading q for which the Interceptor, Moving by Method of Parallel Closing, Intercepts Rectilinearly Flying Target.

1.3.4. Solution of Equations of Motion for Interception by Method of Parallel Closing for a Target Moving Uniformly in a Circle. (v_t = const, ω_t = const) and with Constant Interceptor Velocity (v = const)

Assume that the target flies in a circle at a constant velocity v_t. The interceptor flies toward the target at a constant velocity v by the method of parallel closing. The solution of equations of motion in the polar and Cartesian coordinate system and for certain parameters of the kinematic trajectory are given below.

1. Solution of equations of motion in the polar coordinate system is given by the following formulae.

The distance between the interceptor and target is

$$D = D_0 + \frac{v_t}{\omega_t}(\sin q_0 - \sin q) + \frac{v}{\omega_t} E\left(\frac{1}{k}, q_0\right) - \frac{v}{\omega_t} E\left(\frac{1}{k}, q\right) \quad \text{for} \quad k > 1; \tag{1.55}$$

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$$D=D_0+\frac{v_t}{\omega_t}(\sin q_0-\sin q)+\frac{v}{\omega_t}E(k,\ \varphi_0)-\frac{v}{\omega_t}E(k,\ \varphi)-$$
$$-(1-k^2)F(k,\ \varphi_0)+(1-k^2)F(k,\ \varphi)\quad\text{for}\quad k<1; \tag{1.56}$$

$$D=D_0+2\frac{v_t}{\omega_t}(\sin q_0-\sin q)\qquad\text{for}\quad k=1. \tag{1.57}$$

The target heading is

$$q = q_0 + \omega_t t. \tag{1.58}$$

The bearing is

$$\varphi=\varphi_0+\frac{1}{2}\arcsin\left(1-\frac{2}{k^2}\sin^2 q_0\right)-$$
$$-\frac{1}{2}\arcsin\left[1-\frac{2}{k^2}\sin^2(q_0+\omega_t t)\right], \tag{1.59}$$

where F (k, ϕ) is a first order elliptical integral with modulus k and amplitude ϕ;

E (k, ϕ), E($\frac{1}{k}$, q) is a second order elliptical integral with modulus k($\frac{1}{k}$) and amplitude ϕ(q).

2. The solution of equations of motion in the Cartesian coordinate system is given by the following formulae.

Coordinates of the interceptor:

$$x=x_0+\frac{v_t}{\omega_t}\sin\varepsilon_0\cos(\varepsilon_0-\psi_{t0}-\omega_t t)-\frac{v_t}{\omega_t}\sin\varepsilon_0\cos(\varepsilon_0-\psi_{t0})+$$
$$+\frac{v_t}{\omega_t\sqrt{}}\cos\varepsilon_0 E\left(\frac{1}{k},\ \varepsilon_0-\psi_{t0}\right)-\frac{v}{\omega_t}\cos\varepsilon_0 E\left(\frac{1}{k},\ \varepsilon_0-\psi_{t0}-\omega_t t\right);$$
$$z=z_0-\frac{v_t}{\omega_t}\cos\varepsilon_0\cos(\varepsilon_0-\psi_{t0})+\frac{v_t}{\omega_t}\cos\varepsilon_0\cos(\varepsilon_0-\psi_{t0}-\omega_t t)+$$
$$+\frac{v}{\omega_t}\sin\varepsilon_0 E\left(\frac{1}{k},\ \varepsilon_0-\psi_{t0}\right)-\frac{v}{\omega_t}\sin\varepsilon_0 E\left(\frac{1}{k},\ \varepsilon_0-\psi_{t0}-\omega_t t\right)$$
$$\text{for}\quad k>1; \tag{1.60}$$

$$x=x_0+\frac{v_t}{\omega_t}\sin\varepsilon_0\cos(\varepsilon_0-\psi_{t0}-\omega_t t)-\frac{v_t}{\omega_t}\sin\varepsilon_0\cos(\varepsilon_0-\psi_{t0})+$$
$$+\frac{v}{\omega_t}\cos\varepsilon_0 E(k,\ \varepsilon_0-\psi_{t0})-\frac{v}{\omega_t}\cos\varepsilon_0 E(k,\ \varepsilon_0-\psi_t)+$$
$$+(1-k^2)F(k,\ \varepsilon_0-\psi_t)-(1-k^2)F(k,\ \varepsilon_0-\psi_0);$$

$$z = z_0 - \frac{v_t}{\omega_t}\cos\varepsilon_0\cos(\varepsilon_0 - \psi_{t0}) + \frac{v_t}{\omega_t}\cos\varepsilon_0\cos(\varepsilon_0 - \psi_{t0} - \omega_t t) +$$
$$+ \frac{v}{\omega_t}\sin\varepsilon_0 E(k,\ \varepsilon_0 - \psi_{t0}) - \frac{v}{\omega_t}\sin\varepsilon_0 E(k,\ \varepsilon_0 - \psi_t) +$$
$$+ (1-k^2)F(k,\ \varepsilon_0 - \psi_t) - (1-k^2)F(k,\ \varepsilon_0 - \psi_{t0}) \text{ where } k < 1; \qquad (1.61)$$
$$x = x_0 + \frac{v}{\omega_t}\sin(\psi_{t0} + \omega_t t);$$

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$$z = z_0 - \frac{v}{\omega_t}\cos(\psi_{t0} + \omega_t t),$$
$$\text{where } k = 1, \qquad (1.62)$$

where ε_0 is determined by formula (1.50).

Coordinates of the target are

$$\left.\begin{aligned} x_t &= x_{t0} + \frac{v_t}{\omega_t}\sin(\psi_{t0} + \omega_t t); \\ z_t &= z_m - \frac{v_t}{\omega_t}(\cos\psi_{t0} + \omega_t t). \end{aligned}\right\} \qquad (1.63)$$

Target course is

$$\psi_t = \psi_{t0} + \omega_t t. \qquad (1.64)$$

The distance of the interceptor from the origin of the Cartesian coordinate system is

$$D_t(t) = \sqrt{x^2 + z^2}, \qquad (1.65)$$

where coordinates x and z are calculated from formulae (1.60)-(1.62), (1.64).

3. The radius of curvature of the kinematic trajectory of the interceptor r and the g-force of the interceptor n_r required for flight by this method are determined by the following formulae:

$$r = \frac{v^2\sqrt{1 - \frac{1}{k^2}\sin^2 q}}{v_t\omega_t\cos q} = \frac{v\sqrt{k^2 - \sin^2(q_0 + \omega_t t)}}{\omega_t\cos(q_0 + \omega_t t)} =$$
$$= \frac{v\sqrt{k^2 - \sin^2(\varepsilon_0 + \psi_t)}}{\omega_t\cos(\varepsilon_0 - \psi_t)} = \frac{v\sqrt{k^2 - \sin^2(\varepsilon_0 - \psi_{t0} - \omega_t t)}}{\omega_t\cos(\varepsilon_0 - \psi_{t0} - \omega_t t)}; \qquad (1.66)$$

$$n_r = \frac{v^2}{g_0 r}. \tag{1.67}$$

4. The parameters of motion at the moment of interception (D = 0, x = x_t, z = z_t) are determined by the following expressions.

The target heading q_K from solution of a transcendental equation:

for k > 1

$$\frac{\omega_t D_0}{v_t} \sin q_0 + kE\left(\frac{1}{k}, q_0\right) = \sin q_K + kE\left(\frac{1}{k}, q_K\right); \tag{1.68}$$

for k < 1 /30

$$\frac{\omega_t D_0}{v_t} + \sin q_0 + E(k, \varphi_0) - (1-k^2)F(k, \varphi_0) = \\ = \sin q_K + E(k, \varphi_K) - (1-k^2)F(k, \varphi_K); \tag{1.69}$$

for k = 1

$$\frac{\omega_t D_0}{v_t} + 2\sin q_0 = 2\sin q_K. \tag{1.70}$$

Determination of target heading q_K can be facilitated by using additional charts which are presented in Figures 1.9, 1.10. They contain graphs of the functions in the left and right sides of equations (1.68) and (1.69).

To find the heading q_K one must first calculate the left side of one of the equations (1.68)-(1.69) for the given initial values of magnitude ϕ_0 and $\frac{\omega_t D_0}{v_t}$, and then of target heading q_K from the graphs in Figures 1.9 or 1.10 for the calculated value of the left side at the time of interception.

Example 2. Find the target heading at point of interception q_K, if the target flies in a circle at velocity v_t = 400 m/sec and angular velocity ω_t = -0.05 1/sec. The velocity of the interceptor v = 480 m/sec. The initial positions of the interceptor and target are defined as polar coordinates D_0 = 15,000 m and q_0 = 60°.

The ratio of velocities of the interceptor and target is

$$k = \frac{v}{v_t} = \frac{480}{400} = 1{,}2.$$

Using the graph (see Figure 1.9) for k = 1.2 and $q_0 = 60°$ we determine the value

$$A_0 = \sin q_0 + kE\left(\frac{1}{k},\ q_0\right) = 1{,}95.$$

Then we determine

$$\frac{\omega_t D_0}{v_t} = -\frac{0{,}05 \cdot 15\,000}{400} = -1{,}875.$$

Now we find

$$A = A_0 + \frac{\omega_t D_0}{v_t} = 1{,}95 - 1{,}875 = 0{,}075.$$

Then we find the value $q_K \approx 3°$ from the graph (see Figure 1.9) for k = 1.2 and A = 0.075.

Interception time is:

$$t_к = \frac{q_к - q_0}{\omega_t} = \frac{\psi_{t,к} - \psi_{t0}}{\omega_t}. \tag{1.71}$$

Target bearing is

$$\varphi_к = \varphi_0 + \frac{1}{2}\arcsin\left(1 - \frac{2}{k^2}\sin^2 q_0\right) - \frac{1}{2}\arcsin\left(1 - \frac{2}{k^2}\sin^2 q_к\right). \tag{1.72}$$

5. The zone of attack with respect to g-force. The zone of attack with respect to g-force (see Table 1.1) is determined by the relation /33

$$q \leqslant \arcsin\sqrt{\frac{k^2 - \left(\frac{\omega_t}{\omega_{kn}}\right)^2}{1 - \left(\frac{\omega_t}{\omega_{kn}}\right)^2}} = \arcsin\left[k\sqrt{\frac{1 - \left(\frac{n_t}{n_{kn}}\right)^2}{1 - k^2\left(\frac{n_t}{n_{kn}}\right)^2}}\right]. \tag{1.73}$$

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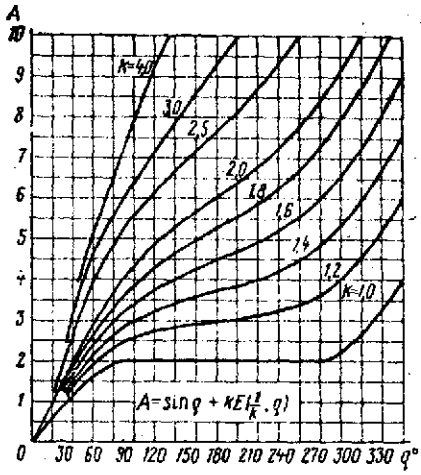


Figure 1.9. Auxiliary Graph for Determination of Target Heading q for $k > 1$.

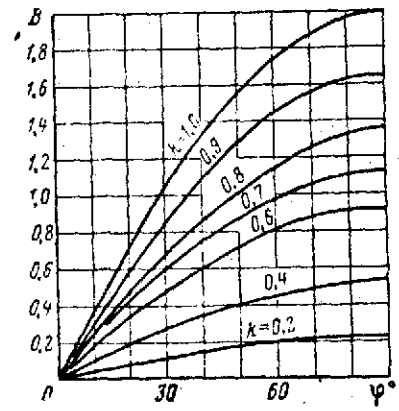
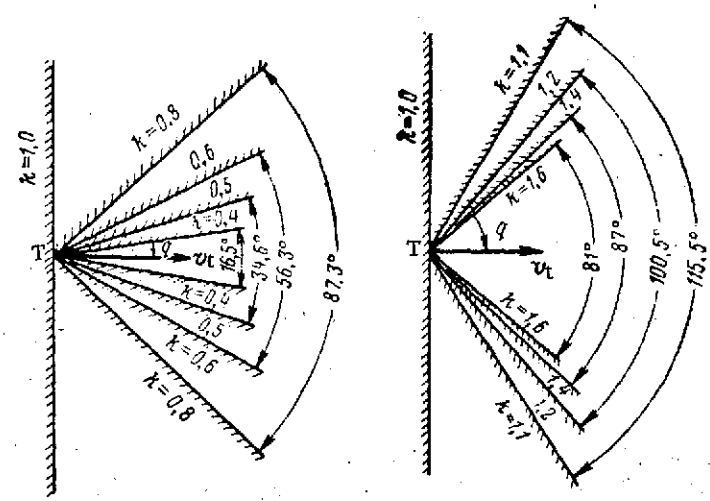


Figure 1.10. Auxiliary Graph for Determination of Target Heading q for $k < 1$:

$B = k \sin \phi + E(k,\phi) - (1 - k^2) \times F(k,\phi)$; ϕ - arcsin (1/k sin q).

/32



a) $\frac{n_t}{n_{kn}} = 0,7$ b) $\frac{n_t}{n_{kn}} = 1,2$

Figure 1.11. Zone of Attack With Respect to g-Force for the Method of Parallel Closing During Guidance of Interceptor to Target Moving in Circle (the Zone of Attack is Located Outside the Hatched Area Bounded by Straight Lines k = const).

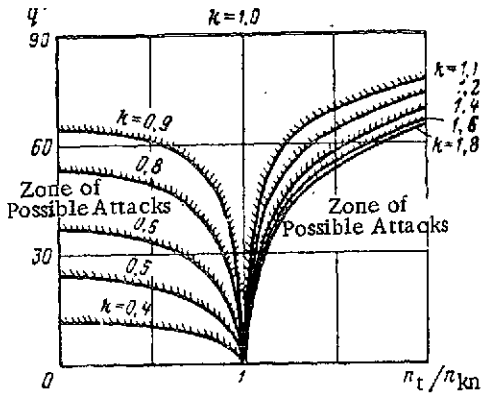


Figure 1.12. Range of Target Headings q, Corresponding to the Zone of Attack With Respect to g-Force, During Guidance Method of Parallel Closing on Target Moving in Circle (Located Outside Hatched Area, Bounded by Straight Lines k = const).

This zone is given in Figure 1.11 for two ratios of target g-force to known interceptor g-force n_t/n_{kn} and different velocity ratios $k = v/v_t$.

The range of target headings q, which corresponds to the zone of attack with respect to g-force, and also the ranges of k, $\frac{n_t}{n_{kn}}$, $\frac{\omega_t}{\omega_{kn}}$, in which this zone exists, are given in Figure 1.12.and in Table 1.7.

1.4. Constant Bearing Guidance Method[3]

The guidance method in which the velocity vector of the interceptor at each moment in time forms a constant angle (bearing) with the "interceptor-target" line is called /34 the *constant target bearing guidance method.* When the bearing is zero ($\phi = 0$), this method coincides with the pursuit curve method. When $\phi = \arcsin(1/k \sin q)$ it coincides with the parallel closing method. The equations of the method are given in Table 1.8.

TABLE 1.7

Velocity Vector $k = \frac{v}{v_t}$	Angular Velocity Ratio $\frac{\omega_t}{\omega_{kn}}$	Load Ratio $\frac{n_t}{n_{kn}}$	Zone of Attack with Respect to g-Force
$k \leqslant 1$	$\left\|\frac{\omega_t}{\omega_{kn}}\right\| \leqslant 1$	$\left\|\frac{n_t}{n_{kn}} - \right\| < \frac{1}{k}$	Present
	$\left\|\frac{\omega_t}{\omega_{kn}}\right\| > 1$	$\left\|\frac{n_t}{n_{kn}}\right\| > \frac{1}{k}$	Not Present

[3] Other terms are also encountered, for example, constant lead attack curve.

TABLE 1.7 (CONTINUED)

Velocity Vector $k = \frac{v}{v_t}$	Angular Velocity Ratio $\frac{\omega_t}{\omega_{kn}}$	Load Ratio $\frac{n_t}{n_{kn}}$	Zone of Attack with Respect to g-Force
$k \geqslant 1$	$\left\lvert \frac{\omega_t}{\omega_{kn}} \right\rvert \geqslant 1$	$\left\lvert \frac{n_t}{n_{kn}} \right\rvert > \frac{1}{k}$	Present
	$\left\lvert \frac{\omega_t}{\omega_{kn}} \right\rvert < 1$	$\left\lvert \frac{n_t}{n_{kn}} \right\rvert < \frac{1}{k}$	Not Present

TABLE 1.8

Coordinate System	Equations of Motion	Equations of Guidance Method
Cartesian	$\dot{x}_t = v_t \cos \psi_t$ $z_t = v_t \sin \psi_t$ $\dot{x} = v \cos \psi$ $\dot{z} = v \sin \psi$	$\frac{x - x_t}{z - z_t} = \operatorname{cotan} (\psi - \phi_0)$
Polar	$\dot{D} = -v \cos \phi_0 - v_t \cos q$ $\dot{q} = \frac{v_t \sin q - v \sin \phi_0}{D} + \omega_t$	$\phi = \phi_0$

1.4.1. Plotting of Kinematic Trajectory of Constant Bearing Method

Suppose that the target moves uniformly at velocity v_t and that the distance D_0 between the interceptor and target and the target heading q_0 are given for the initial moment of time. The bearing angle ϕ_0 is given and is constant throughout the trajectory.

The trajectory of the target is divided into equal segments of length $T_0T_1 = T_1T_2 = T_2T_3 = \ldots = v_t\Delta t$, where Δt is the selected time interval (Figure 1.13). A straight line on which segment $T_0I_0 = D_0$ (initial distance between the interceptor and target) is plotted, is constructed through point T_0 at angle q_0 to line T_0T_1. The straight line, on which segment $I_0I_1 = v\Delta t$ is plotted, is then drawn through point I_0 at angle ϕ_0 to segment I_0T_0. After this a straight line is drawn through point I_1 at angle ϕ_0 to segment I_1T_1. Segment $I_1I_2 = v\Delta t$ is plotted on it. The construction is repeated according to the described plan at point I_2. The resulting broken line $I_0I_1I_2$, ... for $\Delta t \to 0$ is a trajectory with constant bearing.

1.4.2. Equations of Motion and of Constant Bearing Guidance Method

Let the target move arbitrarily at velocity v_t and turn at angular velocity ω_t, whose positive direction is shown in Figure 1.14.

The positions of the interceptor and target in Cartesian coordinate system Oxz, whose origin is arbitrary point O and whose axes are oriented arbitrarily, are defined by the coordinates x, z and x_t, z_t, respectively. The position of the interceptor relative to the target in polar coordinate system TDq, whose origin is connected to the target, is determined by distance D between the target and interceptor and the target heading q, which determines the angular position of the interceptor relative to the target velocity vector. /35

The orientation of velocity vectors of the interceptor v and target v_t in Cartesian coordinate system Oxz is given by angles ψ and ψ_t between the Ox axis and the corresponding vector v or v_t. The orientation of the interceptor velocity vector v in the polar system is given by the target bearing angle ϕ between the "interceptor-target" line and velocity vector v.

The equations of motion of the interceptor and target and the flight conditions of the interceptor according to the constant bearing guidance method, in projections on the axes of the Cartesian and polar coordinate systems, are presented in Table 1.8.

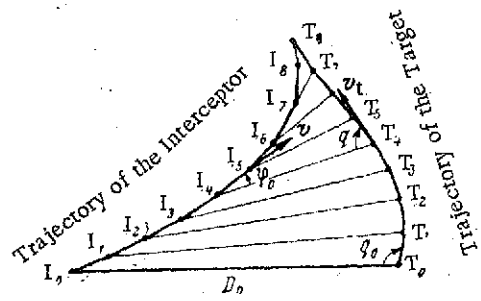


Figure 1.13. Plotting of Kinematic Trajectory in Constant Bearing Method.

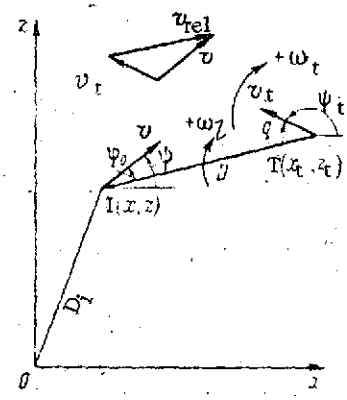


Figure 1.14. Elements of Kinematic Trajectory for Constant Bearing Guidance Method.

Initial conditions. For t = 0 the following coordinates are given:

— in the Cartesian system

$$x_t(0) = x_{t0}; \qquad x(0) = x_0;$$

$$z_t(0) = z_{t0}; \qquad z(0) = z_0;$$

— in the polar system

$$D(0) = D_0; \quad q(0) = q_0.$$

At the same moment of time angle ψ_0 and ϕ_0 are determined by the relations /3

$$\cot(\psi_0 - \psi_0) = \frac{x_0 - x_{t0}}{z_0 - z_{t0}}, \quad \phi = \phi_0.$$

By adding to these conditions the laws of change of target velocity $v_t(t)$, the velocity of the interceptor $v(t)$, the course of target $\psi_t(t)$ or the angular rate of rotation of the target velocity vector $\omega_t(t)$, we obtain a closed system of differential equations in both Cartesian and polar coordinates.

1.4.3. Solution of Equations of Motion for Interception by Constant Bearing Method of a Rectilinearly and Uniformly Flying Target ($\omega_t = 0$, $v_t = \text{const}$) at Constant Interceptor Velocity ($v = \text{const}$)

Suppose that the target flies in a straight line ($\omega_t = \psi_t = 0$) at $v_t = \text{const}$ and the interceptor flies at a constant velocity $v = \text{const}$. The solution of the equations of motion under these conditions for the polar and Cartesian coordinate systems, and for certain characteristics of the kinematic trajectory and zones of attack are given below.

1. The solution of equations of motion in the polar coordinate system is given by the following formulae.

The distance between the interceptor and target:

$$D = D_0 \frac{k\sin\varphi_0 - \sin q_0}{k\sin\varphi_0 - \sin q} \left[\frac{k\sin\varphi_0 \tan\frac{q}{2} - 1 - \sqrt{1-k^2\sin^2\varphi_0}}{k\sin\varphi_0 \tan\frac{q_0}{2} - 1 + \sqrt{1-k^2\sin^2\varphi_0}} \times \right.$$

$$\left. \times \frac{k\sin\varphi_0 \tan\frac{q_0}{2} - 1 + \sqrt{1-k^2\sin^2\varphi_0}}{k\sin\varphi_0 \tan\frac{q}{2} - 1 - \sqrt{1-k^2\sin^2\varphi_0}} \right]^{m_1} = D_0 \left(\frac{k\sin\varphi_0 - \sin q}{k\sin\varphi_0 - \sin q_0} \right)^{m_1 - 1} \times$$

$$\times \left(\frac{1 - k\sin\varphi_0 \sin q_0 + \cos q_0 \sqrt{1-k^2\sin^2\varphi_0}}{1 - k\sin\varphi_0 \sin q + \cos q \sqrt{1-k^2\sin^2\varphi_0}} \right)^{m_1} \qquad (1.74)$$

$$\text{for} \quad k^2\sin^2\varphi_0 < 1;$$

$$D = D_0 \frac{k\sin\varphi_0 - \sin q_0}{k\sin\varphi_0 - \sin q} \exp\left\{ \frac{2k\cos\varphi_0}{\sqrt{k^2\sin^2\varphi_0 - 1}} \times \right.$$

$$\left. \times \left[\arctan \frac{k\sin\varphi_0 \tan\frac{q}{2} - 1}{\sqrt{k^2\sin^2\varphi_0 - 1}} - \arctan \frac{k\sin\varphi_0 \tan\frac{q_0}{2} - 1}{\sqrt{k^2\sin^2\varphi_0 - 1}} \right] \right\}$$

$$\text{for} \quad k^2\sin^2\varphi_0 > 1; \qquad (1.75)$$

$$D = D_0 \frac{1 - \sin q_0}{1 - \sin q} \exp\left\{ k\cos\varphi_0 \left[\tan\left(\frac{q}{2} + \frac{\pi}{4} \right) - \tan\left(\frac{q_0}{2} + \frac{\pi}{4} \right) \right] \right\} \qquad /37$$

$$\text{for} \quad k^2\sin^2\varphi_0 = 1, \qquad (1.76)$$

where

$$m_1 = \frac{k\cos\varphi_0}{\sqrt{1-k^2\sin^2\varphi_0}}. \tag{1.77}$$

The interceptor flight time on the kinematic trajectory from point (D_0, q_0) to point (D, q) is

$$t = -\frac{1}{2}\frac{D_0}{v_t}(1-\sin q_0)\exp\left[-k\cos\varphi_0\tan\left(\frac{\pi}{4}+\frac{q_0}{2}\right)\right]\times$$
$$\times\left\{\left[\frac{1+\tan^2\left(\frac{\pi}{4}+\frac{q}{2}\right)}{k\cos\varphi_0}-\frac{2\tan\left(\frac{\pi}{4}+\frac{q}{2}\right)}{k^2\cos^2\varphi_0}+\frac{2}{k^3\cos^3\varphi_0}\right]\times\right.$$
$$\times\exp\left[k\cos\varphi_0\tan\left(\frac{\pi}{4}+\frac{q}{2}\right)\right]-$$
$$-\left[\frac{1+\tan^2\left(\frac{\pi}{4}+\frac{q_0}{2}\right)}{k\cos\varphi_0}-\frac{2\tan\left(\frac{\pi}{4}+\frac{q_0}{2}\right)}{k^2\cos^2\varphi_0}+\frac{2}{k^3\cos^3\varphi_0}\right]\times$$
$$\left.\times\exp\left[k\cos\varphi_0\tan\left(\frac{\pi}{4}+\frac{q_0}{2}\right)\right]\right\} \quad \text{for} \quad k^2\sin^2\varphi_0 = 1; \tag{1.78}$$

$$t = \frac{1}{v_t(1-k^2)\cos\varphi_0}\{D[k-\cos(q-\varphi_0)]-D_0[k-\cos(q_0-\varphi_0)]\}$$
$$\text{for} \quad k^2\sin^2\varphi_0 \neq 1, \tag{1.79}$$

where

$$\cos\varphi_0 = \frac{\sqrt{k^2-1}}{k}. \tag{1.80}$$

The angular velocity of the "interceptor-target" line is

$$\omega_{\mathcal{L}} = \dot{q} = \frac{v_t}{D_0}\frac{(k\sin\varphi_0-\sin q_0)^{m_1-1}}{(k\sin\varphi_0-\sin q)^{m_1-2}}\left(\frac{1-k\sin\varphi_0\sin q+\cos q\sqrt{1-k^2\sin^2\varphi_0}}{1-k\sin\varphi_0\sin q_0+\cos q_0\sqrt{1-k^2\sin^2\varphi_0}}\right)^m$$
$$\text{for} \quad k^2\sin^2\varphi_0 < 1; \tag{1.81}$$

$$\omega_{\mathcal{L}} = \dot{q} = \frac{v_t}{D_0}\frac{(\sin q-k\sin\varphi_0)^2}{\sin q_0-k\sin\varphi_0}\exp\left\{-\frac{2k\cos\varphi_0}{\sqrt{k^2\sin^2\varphi_0-1}}\times\right.$$
$$\left.\times\left[\arctan\frac{k\sin\varphi_0\tan\frac{q}{2}-1}{\sqrt{k^2\sin^2\varphi_0-1}}-\arctan\frac{k\sin\varphi_0\tan\frac{q_0}{2}-1}{\sqrt{k^2\sin^2\varphi_0-1}}\right]\right\}$$
$$\text{for} \quad k^2\sin^2\varphi_0 > 1. \tag{1.82}$$

When k = 1 the angular velocity of the "interceptor-target" line is finite with the exception of points where the relation /38

$$\sin q = k \sin \phi_0. \tag{1.83}$$

is satisfied.

The angular velocities of the "interceptor-target" line when $\phi_0 < \pi/2$ are the following:

$$\begin{aligned}
&\text{when } m_1 < 2 \quad \omega_l = 0;\\
&\text{when } m_1 > 2 \quad \omega_l = \infty;\\
&\text{when } m_1 = 2 \quad \omega_l = \frac{1}{4}\frac{v_t}{D_0}\left(\frac{1+\sqrt{1-k^2\sin^2\varphi_0}}{k\sin\varphi_0}\right)^2 \times\\
&\times (k\sin\varphi_0 - \sin q_0)(1 - k\sin\varphi_0\sin q_0 -\\
&-\cos q_0\sqrt{1-k^2\sin^2\varphi_0})\, k^4\cos^4\varphi_0,
\end{aligned} \tag{1.84}$$

and in this case $\sin q_0 \neq k \sin \phi_0$.

2. The solution of equations of motion in Cartesian coordinate system is given by the following formulae.

Coordinates of the interceptor:

$$\left.\begin{aligned} x &= x_{t0} + v_t t\cos\psi_{t0} - D\cos(\varphi_0 - \phi);\\ z &= z_{t0} + v_t t\sin\psi_{t0} - D\sin(\varphi_0 - \psi), \end{aligned}\right\} \tag{1.85}$$

where the flight time is determined from formulae (1.87)-(1.89), and the distance D between the interceptor and the target is calculated from formulae (1.74)-(1.76).

The interceptor coordinates for $k^2 \sin^2 \phi_0 < 1$ can also be determined from the relations

$$\begin{aligned}
x = x_{t0} + D_0&\left[\frac{k\sin\varphi_0 + \sin(\psi - \varphi_0 - \psi_{t0})}{k\sin\varphi_0 + \sin(\psi_0 - \varphi_0 - \psi_{t0})}\right]^{m_1-1} \times\\
\times&\left[\frac{k}{1-k^2}\frac{\cos\psi_{t0}}{\cos\varphi_0} + \frac{\cos\psi_{t0}\cos(\psi - 2\varphi_0 - \psi_{t0}) - (1-k^2)\cos(\varphi_0 - \psi)\cos\varphi_0}{(1-k^2)\cos\varphi_0}\right]\times\\
\times&\left[\frac{1 + k\sin\varphi_0\sin(\psi_0 - \varphi_0 - \psi_{t0}) - \cos(\psi_0 - \varphi_0 - \psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}{1 + k\sin\varphi_0\sin(\psi - \varphi_0 - \psi_{t0}) - \cos(\psi - \varphi_0 - \psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}\right]^{m_1} -\\
&- D_0\frac{\cos\psi_{t0}[k + \cos(\psi_0 - \varphi_0 - \psi_{t0})]}{(1-k^2)\cos\varphi_0};
\end{aligned} \tag{1.86}$$

$$z = z_{t0} + D_0 \left[\frac{k\sin\varphi_0 + \sin(\psi-\varphi_0-\psi_{t0})}{k\sin\varphi_0 + \sin(\psi_0-\varphi_0-\psi_{t0})}\right]^{m_1-1} \times$$
$$\times \frac{k\sin\psi_{t0} + \sin\psi_{t0}\cos(\psi-2\varphi_0-\psi_{t0}) - (1-k^2)\sin(\varphi_0-\psi)\cos\varphi_0}{(1-k^2)\cos\varphi_0} \times \qquad (1.86)$$
$$\times \left[\frac{1+k\sin\varphi_0\sin(\psi_0-\varphi_0-\psi_{t0}) - \cos(\psi_0-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}{1+k\sin\varphi_0\sin(\psi-\varphi_0-\psi_{t0}) - \cos(\psi-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}\right]^{m_1} -$$
$$- D_0 \frac{\sin\psi_{t0}[k+\cos(\psi_0-2\varphi_0-\psi_{t0})]}{(1-k^2)\cos\varphi_0}.$$

The flight time of the interceptor on the kinematic trajectory from point (x_0, z_0) to point (x, z) is /39

$$t = \frac{D_0}{v_t(1-k^2)\cos\varphi_0}\left\{\left[\frac{k\sin\varphi_0 + \sin(\psi-\varphi_0-\psi_{t0})}{k\sin\varphi_0 + \sin(\psi_0-\varphi_0-\psi_{t0})}\right]^{m_1-1} \times\right.$$
$$\times [k+\cos(\psi-2\varphi_0-\psi_{t0}]\times$$
$$\times\left[\frac{1+k\sin\varphi_0\sin(\psi_0-\varphi_0-\psi_{t0}) - \cos(\psi_0-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}{1+k\sin\varphi_0\sin(\psi-\varphi_0-\psi_{t0}) - \cos(\psi-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}\right]^{m_1} - k -$$
$$\left. - \cos(\psi_0-2\varphi-\psi_{t0})\right\} \quad \text{for} \quad k^2\sin^2\varphi_0 < 1; \qquad (1.87)$$

$$t = \frac{D_0}{v_t(1-k^2)\cos\varphi_0}\left\{\frac{k\sin\varphi_0 + \sin(\psi_0-\varphi_0-\psi_{t0})}{k\sin\varphi_0 + \sin(\psi-\varphi_0-\psi_{t0})}[k+\cos(\psi-2\varphi_0-\psi_{t0})]\times\right.$$
$$\times\exp\left(\frac{2\cos\varphi_0}{\sqrt{k^2\sin^2\varphi_0-1}}\left[\arctan\frac{k\sin\varphi_0\tan\left(\frac{\psi-\varphi_0-\psi_{t0}}{2}\right)-1}{\sqrt{k^2\sin^2\varphi_0-1}} -\right.\right.$$
$$\left.\left.\left. - \arctan\frac{k\sin\varphi_0\tan\left(\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)-1}{\sqrt{k^2\sin^2\varphi_0-1}}\right]\right) - [k] - \cos(\psi_0-2\varphi_0-\psi_{t0})\Bigg\}$$
$$\text{for} \quad k^2\sin^2\varphi_0 > 1; \qquad (1.88)$$

$$t = -\frac{1}{2}\frac{D_0}{v_t}[1+\sin(\psi_0+\varphi_0-\psi_{t0})]\exp\left[-k\cos\varphi_0\tan\left(\frac{3\pi}{4}+\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)\right]\times$$
$$\times\left\{\left[\frac{1+\tan^2\left(\frac{3\pi}{4}+\frac{\psi-\varphi_0-\psi_{t0}}{2}\right)}{k\cos\varphi_0} - \frac{2\tan\left(\frac{3\pi}{4}+\frac{\psi-\varphi_0-\psi_{t0}}{2}\right)}{k^2\cos^2\varphi_0} + \frac{2}{k^3\cos^3\varphi_0}\right]\times\right.$$
$$\left.\times\exp\left[k\cos\varphi_0\tan\left(\frac{3\pi}{4}+\frac{\psi-\varphi_0-\psi_{t0}}{2}\right)\right]-\right. \qquad (1.89)$$

$$-\left[\frac{1+\tan^2\left(\frac{3\pi}{4}+\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)}{k\cos\varphi_0}-\frac{2\tan\left(\frac{3\pi}{4}+\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)}{k^2\cos^2\varphi_0}+\frac{2}{k^3\cos^3\varphi_0}\right]\times$$
$$\times\exp\left[k\cos\varphi_0\tan\left(\frac{3\pi}{4}+\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)\right]\Bigg\} \quad (1.89)$$
$$\text{for}\quad k^2\sin^2\varphi_0=1.$$

The angular velocity of the "interceptor-target" line

$$\omega_l=\dot{\psi}=\dot{\psi}_0\left[\frac{k\sin\varphi_0+\sin(\psi_0-\varphi_0-\psi_{t0})}{k\sin\varphi_0+\sin(\psi-\varphi_0-\psi_{t0})}\right]^{m_1-2}\times$$
$$\times\left[\frac{1+k\sin\varphi_0\sin(\psi-\varphi_0-\psi_{t0})-\cos(\psi-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}{1+k\sin\varphi_0\sin(\psi_0-\varphi_0-\psi_{t0})-\cos(\psi_0-\varphi_0-\psi_{t0})\sqrt{1-k^2\sin^2\varphi_0}}\right]^{m_1}$$
$$\text{for}\quad k^2\sin^2\varphi_0<1; \quad (1.90)$$

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$$\omega_l=\dot{\psi}=\dot{\psi}_0\left[\frac{k\sin\varphi_0+\sin(\psi-\varphi_0-\psi_{t0})}{k\sin\varphi_0+\sin(\psi_0-\varphi_0-\varphi_{t0})}\right]^2\times$$
$$\times\exp\left\{-\frac{2k\cos\varphi_0}{\sqrt{k^2\sin^2\varphi_0-1}}\left[\arctan\frac{k\sin\varphi_0\tan\left(\frac{\psi-\varphi_0-\psi_{t0}}{2}\right)-1}{\sqrt{k^2\sin^2\varphi_0-1}}-\right.\right.$$
$$\left.\left.-\arctan\frac{k\sin\varphi_0\tan\left(\frac{\psi_0-\varphi_0-\psi_{t0}}{2}\right)-1}{\sqrt{k^2\sin^2\varphi_0-1}}\right]\right\}$$
$$\text{for}\quad k^2\sin^2\varphi_0>1. \quad (1.91)$$

The distance of the interceptor from the origin of the Cartesian coordinate system is

$$D_i=\sqrt{x^2+z^2}, \quad (1.92)$$

where the coordinates of the interceptor x, z are calculated from equation (1.85) or (1.86).

3. The relation of the elements of the kinematic trajectory of the interceptor in the polar and Cartesian coordinate systems is

$$q=\pi+\psi-\varphi_0-\psi_{t0}; \quad (1.93)$$

$$D_0 = \sqrt{(x_0 - x_{t0})^2 + (z_0 - z_{t0})^2}. \tag{1.94}$$

4. The radius r of curvature of the kinematic trajectory of the interceptor and the required g-force n_r for the flight on constant bearing curve are determined by the following formulae:

$$r = \frac{kD}{k\sin\varphi_0 - \sin q} = \frac{kD_0}{k\sin\varphi_0 - \sin q}\left(\frac{k\sin\varphi_0 - \sin q}{k\sin\varphi_0 - \sin q_0}\right)^{m_1 - 1} \times$$
$$\times \left(\frac{1 - k\sin\varphi_0 \sin q_0 + \cos q_0 \sqrt{1 - k^2\sin^2\varphi_0}}{1 - k\sin\varphi_0 \sin q + \cos q \sqrt{1 - k^2\sin^2\varphi_0}}\right)^{m_1} \quad \text{for} \quad k^2\sin^2\varphi_0 < 1; \tag{1.95}$$

$$r = \frac{kD_0(k\sin\varphi_0 - \sin q_0)}{(k\sin\varphi_0 - \sin q)^2} \times$$
$$\times \exp\left[\frac{2k\cos\varphi_0}{\sqrt{k^2\sin^2\varphi_0 - 1}}\left(\arctan\frac{k\sin\varphi_0\tan\frac{q}{2} - 1}{\sqrt{k^2\sin^2\varphi_0 - 1}} - \right.\right.$$
$$\left.\left. - \arctan\frac{k\sin\varphi_0\tan\frac{q_0}{2} - 1}{\sqrt{k^2\sin^2\varphi_0 - 1}}\right)\right] \quad \text{for} \quad k^2\sin^2\varphi_0 > 1; \tag{1.96}$$

$$r = \frac{kD_0(1 - \sin q_0)}{(1 - \sin q)^2}\exp\left\{k\cos\varphi_0\left[\tan\left(\frac{\pi}{4} + \frac{q}{2}\right) - \tan\left(\frac{\pi}{4} + \frac{q_0}{2}\right)\right]\right\} \quad \text{for} \quad k^2\sin^2\varphi_0 = 1, \tag{1.97}$$

/41

where m_1 is determined by formulae (1.77).

$$n_r = \frac{v^2}{g_0 r}. \tag{1.98}$$

5. The parameters of motion at the moment of interception (D = 0, $x = x_t$, $z = z_t$) are represented by the following expressions:

Time of interception

$$t_k = \frac{D_0}{v_t}\,\frac{\cos(q_0 - \varphi_0) - k}{(1 - k^2)\cos\varphi_0} = \frac{D_0}{v_t}\,\frac{\cos(\psi_0 - 2\varphi_0 - \psi_{t0}) - k}{(1 - k^2)\cos\varphi_0} \tag{1.99}$$

for $k^2 \sin^2 \varphi_0 \neq 1$; (1.99)

$$t_\kappa = \frac{1}{2}\frac{D_0}{v_t}(1-\sin q_0)\times$$
$$\times\left[\frac{1+\tan^2\left(\frac{\pi}{4}+\frac{q}{2}\right)}{k\cos\varphi_0}-\frac{2\tan\left(\frac{\pi}{4}+\frac{q}{2}\right)}{k^2\cos^2\varphi_0}+\frac{2}{k^3\cos^3\varphi_0}\right] \quad (1.100)$$
for $k^2 \sin^2 \varphi_0 = 1$.

The square of the interception boundary is

$$R_\kappa^2 = x_{t0}^2 + z_{t0}^2 - 2D_0\frac{x_{t0}\cos\psi_{t0} + z_{t0}\sin\psi_{t0}}{(1-k^2)\cos\varphi_0}[k-\cos(q_0-\varphi_0)] +$$
$$+\frac{D_0^2[k-\cos(q_0-\varphi_0)]^2}{(1-k^2)\cos^2\varphi_0}.$$

The coordinates of the interceptor are

$$x_\kappa = x_{t0} - D_0\frac{\cos\psi_{t0}[k-\cos(q_0-\varphi_0)}{(1-k^2)\cos\varphi_0}; \quad (1.101)$$

$$z_\kappa = z_{t0} - D_0\frac{\sin\psi_{t0}[k-\cos(q_0-\varphi_0)]}{(1-k^2)\cos\varphi_0}; \quad (1.102)$$

$$q_\kappa = \frac{3\pi}{2} \quad \text{for} \quad k^2\sin^2\varphi_0 > 1. \quad (1.103)$$

6. Zone of attack with respect to g-force. The boundary of the zone of attack with respect to g-force (see Table 1.1) is determined by the equation

$$D_{\text{lim}} = \frac{vv_t(k\sin\varphi_0 - \sin q)}{g_0 n_{\text{kn}}}. \quad (1.104)$$

1.5. The Triangulation Method /42

The guidance method in which the interceptor at any moment in time is located on a straight line which connects the guidance center with the target,

is called the triangulation method. The equations of this guidance method are given in Table 1.9. It is assumed that the guidance center is immobile.

1.5.1. Plotting of a Kinematic Trajectory for the Triangulation Method

Let the target fly uniformly at velocity v_t. The target (T) trajectory is divided into equal segments $T_0T_1 = T_1T_2 = T_2T_3 = \ldots = v_t\Delta t$, where Δt is the selected time interval (Figure 1.15). It is assumed that the guidance center is located at point O. Straight lines OT_0, OT_1, OT_2, ... are constructed through the guidance center and points T_0, T_1, T_2, ... For simplicity it is assumed that the interceptor at the initial moment of time is located at point P_0 which coincides with the guidance center. Then the intersection of the straight line segment OT_1 with a circle of radius $v\Delta t$ and center at point I_0, etc., is determined. The broken line $I_0I_1I_2$ ... for $\Delta t \to 0$ gives the trajectory for the triangulation method.

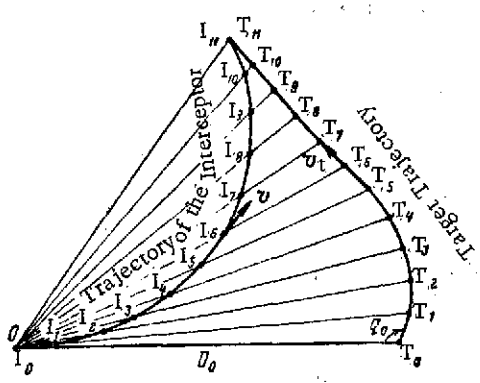

Figure 1.15. Graphic Construction of a Kinematic Trajectory According to the Triangulation Method.

1.5.2. Equations of Motion for Triangulation Guidance Method

Let the target move arbitrarily at velocity v_t and turn at angular velocity ω_t, whose positive direction is shown in Figure 1.16.

The positions of the interceptor and target in Cartesian coordinate system Oxz, whose origin is located at the guidance center, but whose axes are oriented arbitrarily, are defined by the coordinates (x, z) and (x_t, z_t), respectively. The position of the interceptor relative to the target in polar coordinate system TDq, whose pole is directed toward the target, is determined by the distance D between the target and the interceptor and the target heading q which determines the angular position of the interceptor relative to the target velocity vector. The orientation of velocity vectors for interceptor v and target v_t in Cartesian coordinate system Oxz is given by angles ψ and ψ_t between Ox axis and the corresponding vector v or v_t. The orientation of the interceptor velocity vector v in the polar coordinate system is given by the bearing angle ϕ between the "interceptor-target" line and velocity vector v. /43

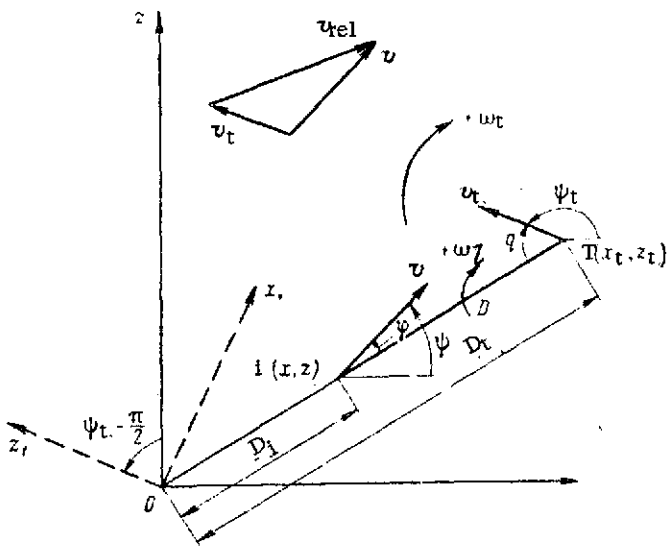

Figure 1.16. Elements of Kinematic Trajectory for the Triangulation Method.

In addition, another Cartesian coordinate system Ox_1z_1 is introduced whose origin is also located at the guidance center O with axis Ox_1 perpendicular to the target velocity vector v_t and axis Oz_1 parallel to it. The coordinates of the interceptor and target in this system are denoted by the index "1": x_1, z_1, x_{1t}, z_{1t}.

Two values are introduced: D_i — the distance from the guidance center to the interceptor and D_t — the distance from the guidance center to the target.

The equations of motion for the interceptor and target and the flight conditions of the interceptor by the triangulation method, written in vector form and in projections on the axes of the two Cartesian and polar coordinate systems, are given in Table 1.9.

Initial conditions. For t = 0 the following coordinates are given:

TABLE 1.9 /44

	Equations of Motion	Equations of Guidance Method
Vectorial form	$v_{rel} = v - v_t - \omega_t \times D$	$v \times D_t - v_t \times D_i = 0$
Cartesian Coordinate System Oxz	$\dot{x}_t = v_t \cos \psi_t$ $\dot{z}_t = v_t \sin \psi_t$ $\dot{x} = v \cos \psi$ $\dot{z} = v \sin \psi$	$\frac{x}{x_t} = \frac{z}{z_t}$

TABLE 1.9 (CONTINUED)

	Equations of Motion	Equations of Guidance Method
Cartesian Coordinate System Ox_1z_1	$\dot{x}_{1t} = 0$ $\dot{z}_{1t} = v_t$ $\dot{x}_1 = v \sin(\psi_t - \psi)$ $\dot{z}_1 = v \cos(\psi_t - \psi)$	$\frac{x_1}{x_{1t}} = \frac{z_1}{z_{1t}}$
Polar Coordinate System	$\dot{D} = -v \cos\phi - v_t \cos q$ $\dot{q} = \frac{v_t \sin q - v \sin\phi}{D} + \omega_t$	$\frac{v_t \sin q - v \sin\phi}{D} = \frac{v_t \sin q}{D_t} = \frac{v \sin\phi}{D_i}$

— in Cartesian system

$$x_t(0) = x_{t0}; \qquad x(0) = x_0;$$

$$z_t(0) = z_{t0}; \qquad z(0) = z_0;$$

— in polar system

$$D(0) = D_0; \qquad q(0) = q_0.$$

At this same moment in time angles ψ_0 and ϕ_0 are determined by the relation

$$\psi_0 = \psi_{t0} + \phi_0 + q_0 - \pi;$$

$$\phi_0 = \arcsin\left(\frac{1}{k}\frac{x_0}{x_{t0}} \sin q_0\right).$$

By adding to these conditions the laws of change of velocity of the target $v_t(t)$, velocity of the interceptor $v(t)$, course of the target $\psi_t(t)$ or the

angular rate of turn of the target velocity vector $\omega_t(t)$, we derive a closed system of differential equations in Cartesian and in the polar coordinates.

1.5.3. Solution of Equations of Motion for Interception by Triangulation of Rectilinearly and Uniformly Flying Target ($\omega_t = 0$, $v_t = \text{const}$) at Constant Interceptor Velocity ($v = \text{const}$) /45

Let the target fly in a straight line ($\omega_t = 0$, $\psi_t = \text{const}$) uniformly ($v_t = \text{const}$), and the interceptor at constant velocity v. The solution of equations of motion under these conditions for Cartesian coordinate systems Oxz and Ox_1z_1, and certain properties of the kinematic trajectory and zone of attack of the interceptor are given below.

1. Solution of equations of motion in the Cartesian coordinate system is given by the following formulae.

Coordinates of the target:

$$x_{1t} = x_{1t_0}; \quad z_{1t} = z_{1t_0} + v_t t. \tag{1.105}$$

Coordinates of the interceptor:

$$x_1 = \sqrt{\frac{\sin\Delta\psi}{\sin\Delta\psi_0}}\, x_{10} - k x_{1t_0} \sqrt{\sin\Delta\psi}\left(\frac{\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}} - \frac{\cos\Delta\psi}{\sqrt{\sin\Delta\psi}}\right) - \\ - \frac{1}{2} k x_{1t_0} \sqrt{\sin\Delta\psi}\,[F(\Delta\psi_0) - F(\Delta\psi)]; \tag{1.106}$$

$$z_1 = z_{10} + \frac{x_{10} - k x_{1t_0}\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}}\left(\frac{\cos\Delta\psi}{\sqrt{\sin\Delta\psi}} - \frac{\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}}\right) - \\ - \left(\frac{x_{10} - k x_{1t_0}\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}} + \frac{k x_{1t_0}\cos\Delta\psi}{2\sqrt{\sin\Delta\psi}}\right)[F(\Delta\psi_0) - F(\Delta\psi)] + \\ + \frac{k x_{1t_0}}{4}[F(\Delta\psi_0) - F(\Delta\psi)]^2 \tag{1.107}$$

or in the system $Oxyz$:

$$\left.\begin{aligned} x &= x_1 \sin\psi_{t_0} + z_1 \cos\psi_{t_0}; \\ z &= -x_1 \cos\psi_{t_0} + z_1 \sin\psi_{t_0}, \end{aligned}\right\} \tag{1.108}$$

where

$$\Delta\psi=\psi_{t0}-\psi;\quad \Delta\psi_0=\psi_{t0}-\psi_0;$$

$$F(\Delta\psi)=\sqrt{2}\,F\left(\frac{\pi}{4},\ \arccos\sqrt{\sin\Delta\psi}\right)-$$

$$-2\sqrt{2}\,E\left(\frac{\pi}{4},\ \arccos\sqrt{\sin\Delta\psi}\right);$$

$F\left(\frac{\pi}{4},\ \arccos\sqrt{\sin\Delta\psi}\right)$ is the first order elliptical integral with modulus $\frac{\pi}{4}$ and argument $\arccos\sqrt{\sin\Delta\psi}$; $E\left(\frac{\pi}{4},\ \arccos\sqrt{\sin\Delta\psi}\right)$ is the second order elliptical integral with modulus $\frac{\pi}{4}$ and argument $\arccos\sqrt{\sin\Delta\psi}$. /46

The interceptor flight time on the kinematic trajectory from point $(x_0,\ z_0)$ to point $(x,\ z)$ is

$$t=-\frac{x_{1t0}}{v_t}\left(1-\sqrt{\frac{\sin\Delta\psi_0}{\sin\Delta\psi}}\right)+$$

$$+\frac{x_{1t0}}{2v_t\sqrt{\sin\Delta\psi}}[F(\Delta\psi)-F(\Delta\psi_0)]=\frac{D_{t0}}{v_t}\sin q_0(\cot q_0-\cot q). \tag{1.109}$$

The angular velocity of the "interceptor-target" line (either the "guidance center-interceptor" line, or the "guidance center-target" line) is

$$\omega_l=\frac{v_t}{x_{1t0}}\times$$

$$\times\frac{1}{1+\frac{1}{\sin\Delta\psi_0\sin\Delta\psi}\left\{\frac{x_{10}}{kx_{1t0}}-\cos\Delta\psi_0-\frac{\sqrt{\sin\Delta\psi_0}}{2}[F(\Delta\psi_0)-F(\Delta\psi)]\right\}^2}. \tag{1.110}$$

The distance from the guidance center to the interceptor is

$$D_i=x_{1t0}\sqrt{1+\frac{1}{k^2\sin\Delta\psi_0\sin\Delta\psi}\left\{\frac{x_{10}}{x_{1t0}}-k\cos\Delta\psi_0-\right.}$$

$$\left.\sqrt{-\frac{k\sqrt{\sin\Delta\psi_0}}{2}[F(\Delta\psi_0-F(\Delta\psi)]}\right\}^2}. \tag{1.111}$$

. The distance from the guidance center to the target is

$$D_t = x_{1t0}\sqrt{\frac{\sin\Delta\psi}{\sin\Delta\psi_0}}\left\{\frac{x_{10}}{x_{1t0}} - k\sqrt{\sin\Delta\psi_0}\left(\frac{\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}} - \frac{\cos\Delta\psi}{\sqrt{\sin\Delta\psi}}\right) - \right.$$
$$\left. - \frac{k\sqrt{\sin\Delta\psi_0}}{2}[F(\Delta\psi_0) - F(\Delta\psi)]\right\} \times$$
$$\times\left[1 + \frac{1}{k^2\sin\Delta\psi_0\sin\Delta\psi}\left\{\frac{x_{10}}{x_{1t0}} - k\cos\Delta\psi_0 - \right.\right.$$
$$\left.\left. - \frac{k\sqrt{\sin\Delta\psi_0}}{2}[F(\Delta\psi_0) - F(\Delta\psi)]\right\}^2\right]^{\frac{1}{2}}. \qquad (1.112)$$

The values of the function $F(\Delta\psi)$ are given in Table 1.10 and in Figure 1.17.

2. The relation of the elements of the kinematic trajectory of the interceptor in the polar and Cartesian coordinate systems is /47

$$D = \sqrt{(x_1 - x_{1t})^2 + (z_1 - z_{1t})^2} = \frac{x_{1t} - x_1}{\sin q} =$$
$$= x_{1t0}\sqrt{\frac{\sin\Delta\psi}{\sin\Delta\psi_0}}\left\{\sqrt{\frac{\sin\Delta\psi_0}{\sin\Delta\psi}}(1 - k\cos\Delta\psi) - \frac{x_{10}}{x_{1t0}} + k\cos\Delta\psi_0 + \right.$$
$$\left. + \frac{k}{2}\sqrt{\sin\Delta\psi_0}[F(\Delta\psi_0) - F(\Delta\psi)]\right\} \times$$
$$\times\left[1 + \frac{1}{k^2\sin\Delta\psi_0\sin\Delta\psi}\left\{\frac{x_{10}}{x_{1t0}} - k\cos\Delta\psi_0 - \right.\right.$$
$$\left.\left. - \frac{k\sqrt{\sin\Delta\psi_0}}{2}[F(\Delta\psi_0) - F(\Delta\psi)]\right\}^2\right]^{\frac{1}{2}}. \qquad (1.113)$$

The relation between the angles is

$$\phi + q = \pi + \psi - \psi_t; \qquad (1.114)$$

$$\cot\phi = (\cos\Delta\psi - k\frac{x_{1t}}{x_1}\operatorname{cosec}\Delta\psi; \qquad (1.115)$$

$$\cot q = \left(\cos\Delta\psi - \frac{1}{k}\frac{x_1}{x_{1t}}\right)\operatorname{cosec}\Delta\psi =$$
$$= \sqrt{\frac{\sin\Delta\psi_0}{\sin\Delta\psi}}\left(\frac{1}{k}\frac{x_{10}}{x_{1t0}} - \cos\Delta\psi_0\right)\operatorname{cosec}\Delta\psi_0 +$$
$$+ \frac{1}{2\sqrt{\sin\Delta\psi}}[F(\Delta\psi) - F(\Delta\psi_0)]; \qquad (1.116)$$

$$\sin q = \left\{1 + \frac{1}{\sin\Delta\psi_0 \sin\Delta\psi}\left[\frac{x_{10}}{kx_{1t0}} - \cos\Delta\psi_0 - \frac{\sqrt{\sin\Delta\psi_0}}{2}\left(F(\Delta\psi_0) - F(\Delta\psi)\right)\right]^2\right\}^{-\frac{1}{2}}. \quad (1.117)$$

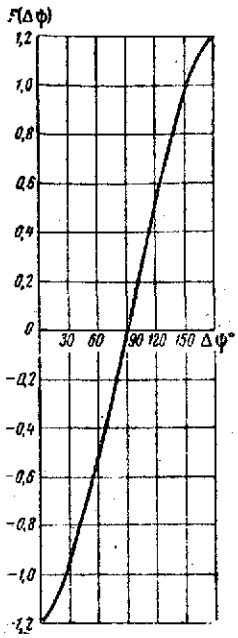

Figure 1.17. Graph of the Function $F(\Delta\psi) = \sqrt{2}F\left(\frac{\pi}{4}, \arccos\sqrt{\sin\Delta\psi}\right) - 2\sqrt{2}E\left(\frac{\pi}{4}, \arccos\sqrt{\sin\Delta\psi}\right)$

where

3. The radius of curvature r of the kinematic trajectory of the interceptor and the required g-force n_r for the flight on the kinematic trajectory by the triangulation method are determined by the following formulae:

$$r = \frac{x_{1t0}\cot\Delta\psi}{2\sqrt{\sin\Delta\psi}}\left\{\frac{x_{10}}{x_{1t0}\sqrt{\sin\Delta\psi_0}} - k\left(\frac{\cos\Delta\psi_0}{\sqrt{\sin\Delta\psi_0}} - \frac{\cos\Delta\psi}{\sqrt{\sin\Delta\psi}}\right) - \frac{k}{2}\left[F(\Delta\psi_0) - F(\Delta\psi)\right]\right\}; \quad (1.118)$$

$$n_r = \frac{v^2}{g_0 r} \quad (1.119)$$

4. The parameters of motion at the time of interception (D = 0, x = x_t, z = z_t) are represented by the following expressions. /51

The angle $\Delta\psi_K$ is determined from the solution of the transcendental equation

$$S_к(k, \Delta\psi_к) = S_{к0}\left(\frac{x_{10}}{kx_{1t0}}, \Delta\psi_0\right), \quad (1.120)$$

$$S_к(k, \Delta\psi_к) = 2\frac{\cos\Delta\psi_к - \frac{1}{k}}{\sqrt{\sin\Delta\psi_к}} + F(\Delta\psi_к). \quad (1.121)$$

$$S_{к0}\left(\frac{x_{10}}{kx_{1t0}}, \Delta\psi_0\right) = 2\frac{\cos\Delta\psi_0 - \frac{x_{10}}{kx_{1t0}}}{\sqrt{\sin\Delta\psi_0}} + F(\Delta\psi_0); \quad (1.122)$$

$$\Delta\psi_x = \psi_x - \psi_{tx}. \tag{1.123}$$

For solution of equations (1.120) it is helpful to use the graphs on Figures 1.18 and 1.19 or Table 1.10. The sequence for determination of $\Delta\psi_K$ is the following. Based on the given initial conditions for x_{10}, x_{1t0}, k, $\Delta\psi_0$ the value of S_{K0} is found using the graph in Figure 1.18. This value may be more accurately determined from Table 1.10. For this purpose we first find in the table the values of S and S_K (points which fall in the interval between $\Delta\psi$ or k given in the table are calculated by means of linear interpolation) for the given $\Delta\psi_0$ and k. Then, having calculated the difference $\Delta S = S_K - S$, we multiply it by the initial value x_{10}/x_{1t0}: $\Delta S(x_{10})/x_{1t0}$. The result is added to S_K, giving S_{K0}: $S_{K0} = S_K + \Delta S(x_{10}/x_{1t0})$. $\Delta\psi_K$ is found for the calculated S_{K0} and given k according to the graph in Figure 1.19. The value of $\Delta\psi_K$ is further refined using Table 1.10. To do this we find the S_K in the table for the given k which is equal to S_{K0} (if necessary, we carry out linear interpolation). The $\Delta\psi$, which corresponds to this value of $S_K = S_{K0}$, is $\Delta\psi_K$.

Example 3. An interceptor flying at velocity v = 480 m/sec, is guided to a target which is flying rectilinearly at velocity v_t = 400 m/sec (k = = 1.2). Find the angle $\Delta\psi_K$ at the moment of interception if at the initial moment in time we are given: $\Delta\psi_0 = 60°$, $x_{10}/x_{1t0} = 0.12$. First we determine the value of S_{K0}: from the graph in Figure 1.18 $S_{K0} = 0.35$; from Table 1.10:

for k = 1.1 and $\Delta\psi_0 = 60°$ ..... $S_K = 1.3904$;

for k = 1.3 and $\Delta\psi_0 = 60°$ ..... $S_K = 1.0898$.

After linear interpolation for k = 1.2 and $\Delta\psi_0 = 60°$, $S_K = -1.2401$.

For $\Delta\psi_0 = 60°$ we obtain S = 0.5634. Hence $\Delta S = S_K - S$ = 01.2401 - 0.5634 = /54 = 1.8035. $\Delta S(x_{10}/x_{1t0}) = -1.8035 \times 0.12 = -0.2164$. Then $S_{K0} = S_K +$ $+ \Delta S(x_{10}/x_{1t0}) = 0.5634 - 0.2164 = 0.3470$.

TABLE 1.10. VALUES OF FUNCTIONS F ($\Delta\psi$), S, S_K (k, $\Delta\psi$)

$\Delta\psi$ Course Degree	$F(\Delta\psi)$	$S = F(\Delta\psi) + \frac{2\cos\Delta\psi}{\sqrt{\sin\Delta\psi}}$	$S_K = F(\Delta\psi) + 2\frac{\cos\Delta\psi - \frac{1}{\psi}}{\sqrt{\sin\Delta\psi}}$							
			$k=0,5$	$k=0,7$	$k=0,9$	$k=1,0$	$k=1,1$	$k=1,3$	$k=1,5$	$k=2,0$
1	—1,1970	13,9171	—16,3172	—7,6793	—2,8796	—1,2001	0,1741	2,2890	3,8385	6,3585
2	—1,1937	9,5065	—11,9068	—5,7890	—2,3897	—1,2001	—0,2269	1,2710	2,3680	4,1532
3	—1,1900	7,5428	—9,9474	—4,9504	—2,1739	—1,2023	—0,4074	0,8161	1,7124	3,1702
4	—1,1858	6,3661	—8,7739	—4,4484	—2,0450	—1,2039	—0,5158	0,5433	1,3192	2,5811
5	—1,1809	5,5661	—7,9794	—4,1095	—1,9591	—1,2067	—0,5911	0,3465	1,0507	2,1792
6	—1,1757	4,9784	—7,3978	—3,8619	—1,8972	—1,2097	—0,6472	0,2185	0,8528	1,8844
7	—1,1696	4,5170	—6,9410	—3,6675	—1,8485	—1,2120	—0,6912	0,1103	0,6975	1,6525
8	—1,1633	4,1466	—6,5773	—3,5135	—1,8110	—1,2153	—0,7279	0,0222	0,5718	1,4656
9	—1,1567	3,8380	—6,2758	—3,3863	—1,7807	—1,2188	—0,7591	—0,0517	0,4666	1,3096
10	—1,1493	3,5774	—6,0218	—3,2793	—1,7555	—1,2222	—0,7859	—0,1145	0,3775	1,1776
15	—1,1090	2,6885	—5,1747	—2,9282	—1,6799	—1,2431	—0,8857	—0,3357	0,0673	0,7227
20	—1,0612	2,1525	—4,6874	—2,7333	—1,6474	—1,2675	—0,9566	—0,4782	—0,1276	0,4425
25	—1,0073	1,7809	—4,3720	—2,6141	—1,6373	—1,2955	—1,0159	—0,5855	—0,2701	0,2427
30	—0,9481	1,5013	—4,1556	—2,5394	—1,6414	—1,3231	—1,0700	—0,6743	—0,3844	0,0671
35	—0,8840	1,2792	—4,0020	—2,4932	—1,6548	—1,3614	—1,1214	—0,7519	—0,4813	—0,0411
40	—0,8060	1,1049	—3,8845	—2,4590	—1,6670	—1,3898	—1,1630	—0,8140	—0,5583	—0,1424
45	—0,7441	0,9377	—3,8191	—2,4601	—1,7049	—1,4407	—1,2245	—0,8918	—0,6480	—0,2515
50	—0,6695	0,7994	—3,7710	—2,4652	—1,7397	—1,4858	—1,2781	—0,9584	—0,7241	—0,3432
55	—0,5914	0,6762	—3,7437	—2,4809	—1,7793	—1,5337	—1,3328	—1,0237	—0,7971	—0,4288
60	—0,5112	0,5634	—3,7349	—2,5069	—1,8245	—1,5858	—1,3904	—1,0898	—0,8695	—0,5112
65	—0,4292	0,4586	—3,7431	—2,5427	—1,8756	—1,6422	—1,4512	—1,1573	—0,9420	—0,5918
70	—0,3448	0,3608	—3,7655	—2,5866	—1,9315	—1,7023	—1,5148	—1,2261	—1,0147	—0,6708
75	—0,2605	0,2662	—3,8038	—2,6410	—1,9949	—1,7688	—1,5838	—1,2991	—1,0905	—0,7513
80	—0,1738	0,1761	—3,8545	—2,7029	—2,0631	—1,8392	—1,6560	—1,3740	—1,1675	—0,8316
85	—0,0870	0,0877	—3,9199	—2,7749	—2,1387	—1,9161	—1,7339	—1,4536	—1,2482	—0,9142
90	0,0000	0,0000	—4,0000	—2,8572	—2,2222	—2,0000	—1,8182	—1,5384	—1,3334	—1,0000
95	0,0870	—0,0877	—4,0953	—2,9503	—2,3141	—2,0915	—1,9093	—1,6290	—1,4236	—1,0896
100	0,1738	—0,1761	—4,2067	—3,0551	—2,4153	—2,1914	—2,0082	—1,7263	—1,5197	—1,1838
105	0,2605	—0,2662	—4,3362	—3,1734	—2,5273	—2,3012	—2,1162	—1,8315	—1,6229	—1,2837
110	0,3448	—0,3608	—4,4871	—3,3082	—2,6531	—2,4239	—2,2364	—1,9477	—1,7363	—1,3924
115	0,4292	—0,4586	—4,6603	—3,4599	—2,7928	—2,5594	—2,3684	—2,0745	—1,8592	—1,5090
120	0,5112	—0,5634	—4,8617	—3,6337	—2,9513	—2,7126	—2,5172	—2,2166	—1,9963	—1,6380
125	0,5914	—0,6762	—5,0961	—3,8333	—3,1317	—2,8861	—2,6852	—2,3761	—2,1495	—1,7812
130	0,6695	—0,7994	—5,3698	—4,0640	—3,3385	—3,0846	—2,8769	—2,5572	—2,3229	—1,9420
135	0,7441	—0,9377	—5,6945	—4,3355	—3,5803	—3,3161	—3,0999	—2,7672	—2,5234	—2,1269
140	0,8060	—1,1049	—6,0943	—4,6688	—3,8768	—3,5996	—3,3728	—3,0238	—2,7681	—2,3522
145	0,8840	—1,2792	—6,5604	—5,0516	—4,2132	—3,9198	—3,6798	—3,3103	—3,0397	—2,5995
150	0,9481	—1,5013	—7,1582	—5,5420	—4,6440	—4,3297	—4,0726	—3,6769	—3,3870	—2,9155

Commas indicate decimal points

TABLE 1.10 (CONTINUED)

$\Delta\psi$ Course Degree	$F(\Delta\psi)$	$S=F(\Delta\psi)+\frac{2\cos\Delta\psi}{\sqrt{\sin\Delta\psi}}$	$S_K=F(\Delta\psi)+2\frac{\cos\Delta\psi-\frac{1}{k}}{\sqrt{\sin\Delta\psi}}$							
			$k=0,5$	$k=0,7$	$k=0,9$	$k=1,0$	$k=1,1$	$k=1,3$	$k=1,5$	$k=2,0$
155	1,0073	—1,7809	—7,9338	—6,1759	—5,1991	—4,8573	—4,5777	—4,1473	—3,8319	—3,3191
160	1,0612	—2,1525	—8,9924	—7,0383	—5,9524	—5,5725	—5,2616	—4,7832	—4,4326	—3,8625
165	1,1090	—2,6885	—10,5517	—8,3052	—7,0568	—6,6201	—6,2627	—5,7127	—5,3097	—4,6543
170	1,1493	—3,5774	—13,1766	—10,4341	—8,9103	—8,3770	—7,9407	—7,2692	—6,7773	—5,9772
171	1,1567	—3,8380	—13,9518	—11,0623	—9,4567	—8,8948	—8,4351	—7,7277	—7,2094	—6,3664
172	1,1633	—4,1466	—14,8705	—11,8067	—10,1042	—9,5085	—9,0211	—8,2710	—7,7214	—6,8276
173	1,1696	—4,5170	—15,9750	—12,7015	—10,8825	—10,2460	—9,7252	—8,9237	—8,3365	—7,3815
174	1,1757	—4,9784	—17,3546	—13,8187	—11,8540	—11,1665	—10,6040	—9,7383	—9,1040	—8,0724
175	1,1809	—5,5661	—19,1116	—15,2417	—13,0913	—12,3389	—11,7233	—10,7757	—10,0815	—8,9525
176	1,1858	—6,3661	—21,5061	—17,1806	—14,7772	—13,9361	—13,2480	—12,1889	—11,4130	—10,1511
177	1,1900	—7,5428	—25,0330	—20,0360	—17,2595	—16,2879	—15,4930	—14,2695	—13,3732	—11,9154
178	1,1937	—9,5067	—30,9198	—24,8020	—21,4027	—20,2131	—19,2399	—17,7420	—16,6446	—14,8598
179	1,1970	—13,9171	—44,1514	—35,5135	—30,7138	—29,0343	—27,6601	—25,5452	—23,9957	—21,4757

Commas indicate decimal points

Then $\Delta\psi_K$ is found from the known values of $S_K = S_{K0}$ and k: from the graph in Figure 1.19 $\Delta\psi_K \approx 2°.5$; from Table 1.10 for k = 1.1 and $\Delta\psi = 2°$, $S_K = -0.2269$; for k = 1.3 and $\Delta\psi = 2°$, $S_K = 1.2710$; for k = 1.1 and $\Delta\psi = 3°$, $S_K = -0.4074$; for k = 1.3 and $\Delta\psi = 3°$, $S_K = 0.8161$. S_K for k = 1.2 and $\Delta\psi = 2°$ is calculated by means of linear interpolation and is $S_K = 0.5220$; for k = 1.2 and $\Delta\psi = 3°$, $S_K = 0.2043$. Now it is easy to find the $\Delta\psi_K$, which corresponds to k = 1.2 and $S_{K0} = 0.3470$. This value $\Delta\psi_K$ is calculated by linear interpolation and is $\Delta\psi_K \approx 2°.55$.

The target heading and bearing (Figure 1.20) are:

$$q_K = \arctan\left(\cot\Delta\psi_K - \frac{1}{k\sin\Delta\psi_K}\right); \tag{1.124}$$

$$\phi_K = \arctan\left(\cot\Delta\psi_K - \frac{k}{\sin\Delta\psi_K}\right). \tag{1.125}$$

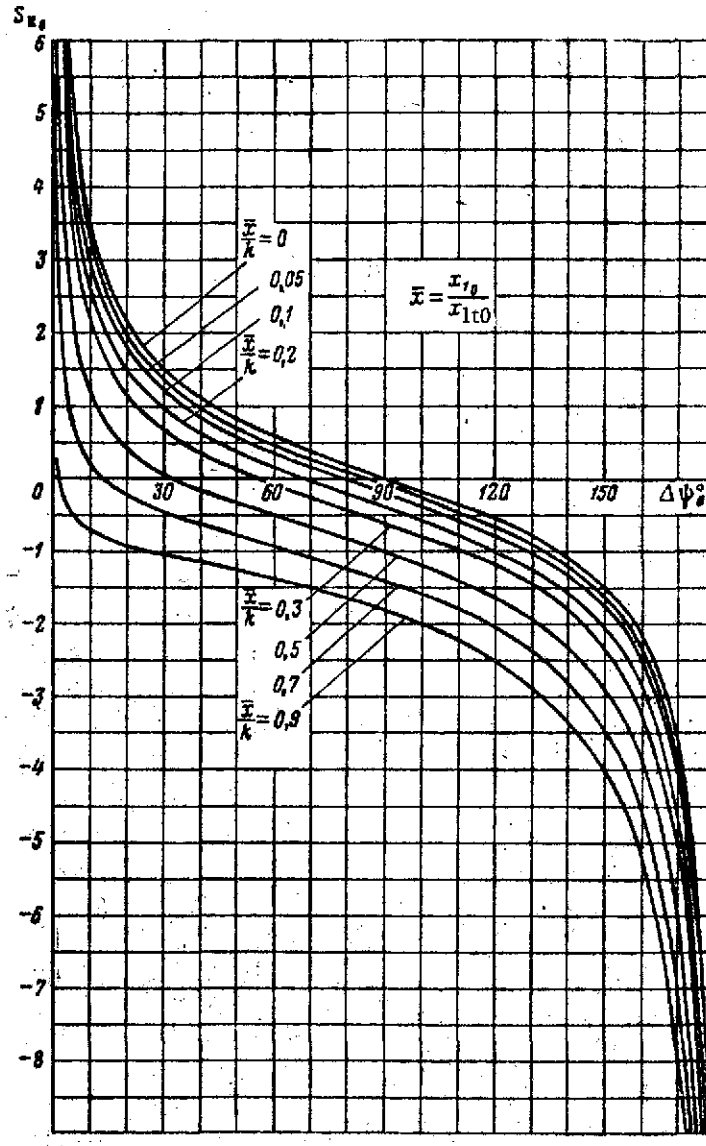

Figure 1.18. Graph of the Function

$$S_{\Pi\Phi}\left(\Delta\psi_0, \frac{\bar{x}}{k}\right) = F(\Delta\psi) + 2\frac{\cos\Delta\psi_0 - \frac{\bar{x}}{k}}{\sqrt{\sin\Delta\psi}}$$

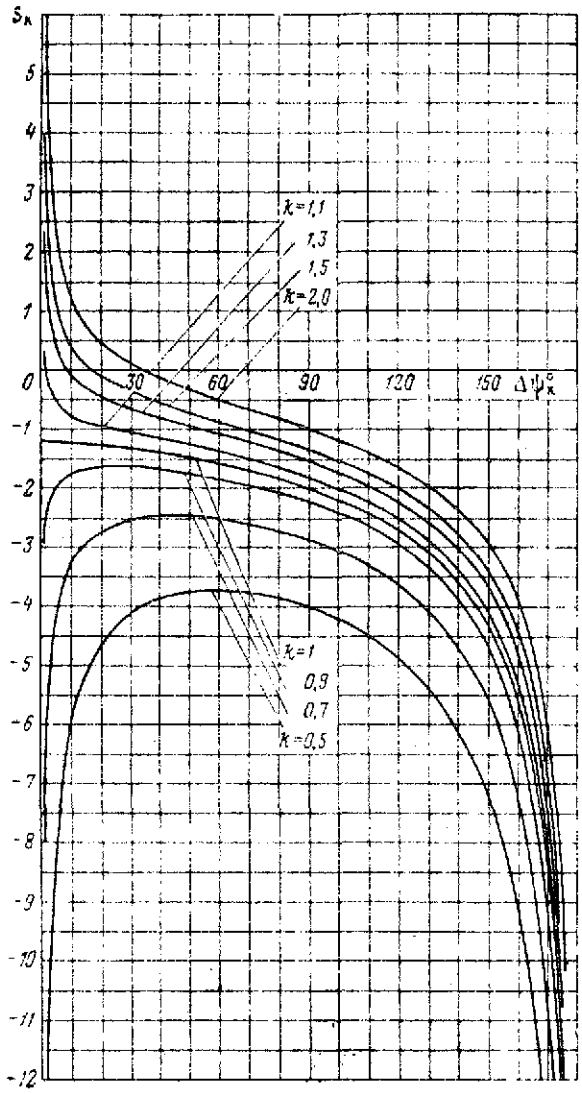

Figure 1.19. Graph for Determining Angle $\Delta\psi_K$ at Moment of Interception

The graph of the functions q_K and ϕ_K are given in Figure 1.20.

The coordinates of the interceptor and target are: /56

$$\left.\begin{aligned} x_{1t.\kappa} &= x_{1\kappa} = x_{1t0}, \\ z_{1t.\kappa} &= z_{1\kappa} = x_{1t0}\frac{k\cos\Delta\psi_\kappa - 1}{k\sin\Delta\psi_\kappa} = x_{1t0}\lambda(k, \Delta\psi_\kappa). \end{aligned}\right\} \quad (1.126)$$

The graph of the function $\lambda(k, \Delta\psi_K)$ from (1.126) is represented in Figure 1.21.

The time of interception is $t_\kappa = \frac{D_{t0}\sin q_0}{v_t}(\cot q_0 - \cot q_\kappa) = \frac{x_{1t0}}{v_t} \times$

$$\times\left(\cot\Delta\psi_0 - \cot\Delta\psi_\kappa - \frac{x_{10}}{kx_{1t.0}\sin\Delta\psi_0} + \right.$$

$$\left. + \frac{1}{k\sin\Delta\psi_\kappa}\right). \quad (1.127)$$

The boundary of interception is

$$R_\kappa = \frac{x_{1t0}}{k\sin\Delta\psi_\kappa}\sqrt{1 + k^2 - 2k\cos\Delta\psi_\kappa}. \quad (1.128)$$

5. Zone of attack with respect to velocity. When the interceptor velocity is greater than the velocity of the target ($k > 1$), the zone of attack with respect to velocity (see Table 1.1) represents the entire guidance plane. When the interceptor velocity is less than or equal to the target velocity ($k \leqslant 1$), the zone of attack with respect to velocity is limited. The range of initial angles $\Delta\psi_0^*$, at which the interceptor begins motion and intercepts the target, is determined by the equation

$$2\cdot\frac{k\cos\Delta\psi_0^* - \frac{x_{10}}{x_{1t0}}}{k\sqrt{\sin\Delta\psi_0^*}} + F(\Delta\psi_0^*) = F(k) + 2\frac{k^2 - 1}{k\sqrt[4]{1 - k^2}}. \quad (1.129)$$

Figure 1.22 shows the zone of interception based on initial angles $\Delta\psi_0^*$ and velocity ratio k. The values of initial angles $\Delta\psi_0^*$, which correspond to equation (1.129), are given in Table 1.11.

6. The zone of attack with respect to g-force. The boundary of the zone of attack with respect to g-force (see Table 1.1) is determined by equation /57

$$\left(\frac{D}{D_t}\right)_{\lim} = 1 - \frac{k}{\sin q}\left[1 + \left(\frac{\cos q}{\frac{g_0 n_{km} D_t}{2v v_t} - \sin q}\right)^2\right]^{-1}. \qquad (1.130)$$

When $\left(\frac{\cos q}{\frac{g_0 n_{km} D_t}{2v v_t} - \sin q}\right)^2 \ll 1$ the shape of the boundary curve is simplified:

$$\left(\frac{D}{D_t}\right)_{\lim} = 1 - \frac{k}{\sin q}. \qquad (1.130')$$

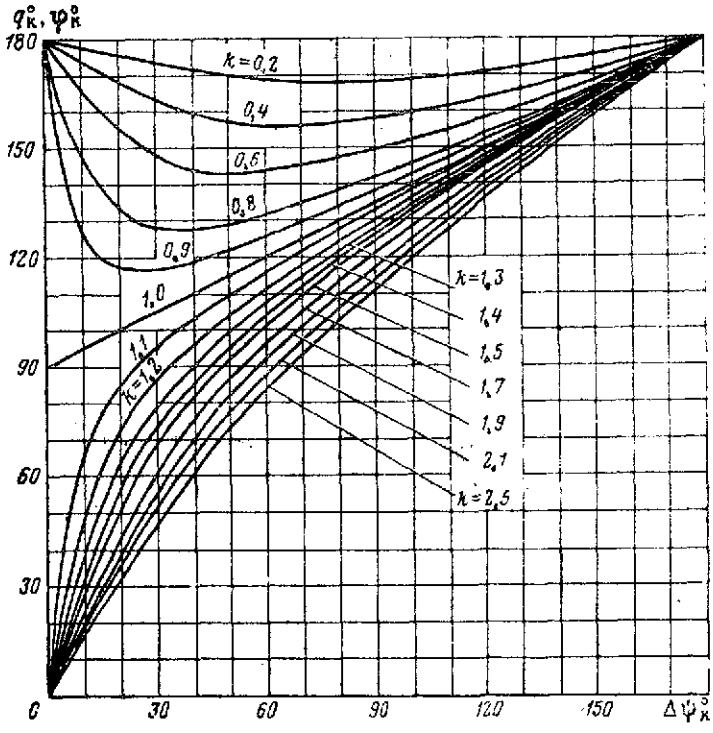

Figure 1.20. Graphs of the Functions $q_K(k, \Delta\psi_K)$ and $\phi_K(k_1, \Delta\psi_K)$.

This approximate zone of attack by overload is given in Figure 1.23 for target headings $0° \leqslant q \leqslant 180°$. The zone of attack in Figure 1.23 is plotted in coordinates $q, (D/D_t)_{lim}$, i.e., the ratios of the distance between the target and interceptor to the distance from the guidance center to the target correspond to the rays coming from point T. The rays go at angles equal to the corresponding target headings q. Here the locations of the interceptor, target and guidance center given in Figure 1.16 correspond to target headings $0 \leqslant q \leqslant 180°$; the following mutual positions of guidance center, interceptor and target correspond to headings $180° \leqslant q \leqslant 360°$: /58

— the target and interceptor are on different sides of the guidance center (Figure 1.24, a);

— the target is between the interceptor and the guidance center (Figure 1.24, b).

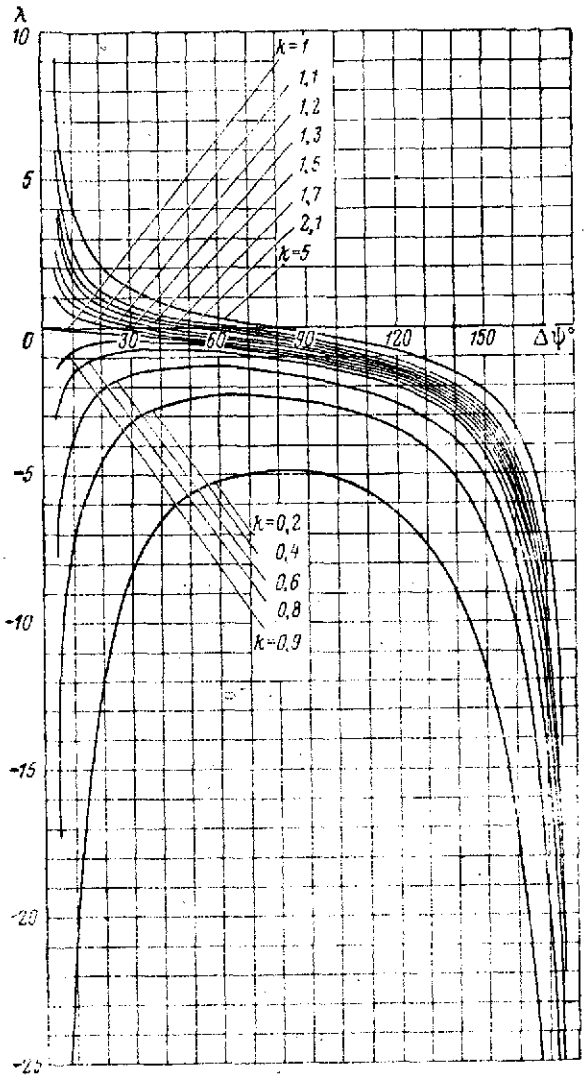

Figure 1.21. Graph of the Function $\lambda = \cot \Delta\psi_K - 1/(k \sin \Delta\psi_K)$.

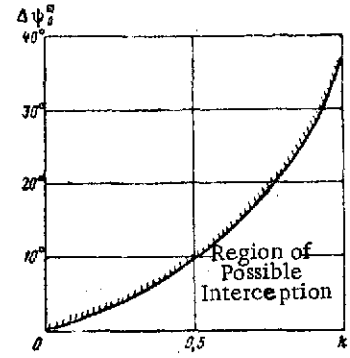

Figure 1.22. Zone of Interception with Respect to Initial Angle $\Delta\psi_0^*$ and Velocity Ratio k for Triangulation Method.

The dimensions of the zone of attack with respect to g-force are given in Table 1.12.

It should be noted that part /59 of the zone of attack with respect to g-force when $k < 1$ and $(D/D_t)_{lim} > 0$, which represents the area of the guidance plane bounded by curve (1.130), is located outside the zone of attack with respect to velocity, i.e., the curve (1.130) in Figure 1.23 serves as a boundary between:

— the region where the interception is impossible due to insufficient g-force in the interceptor for flight on a kinematic trajectory;

— the region where interception is impossible because the target flies at greater velocity than the interceptor (the interceptor falls behind the target and cannot overtake it).

1.5.4. Solution of Equations of Motion for Interception by Triangulation of a Rectilinearly and Uniformly Flying Target ($\omega_t = 0$, $v_t = \text{const}$) with Variable Interceptor Velocity ($v = \text{var}$)

Let the target fly in a straight line ($\omega_t = 0$; $v_t = \text{const}$) uniformly ($v_t = \text{const}$). The interceptor flies at variable velocity ($v = \text{var}$). The

solution of equations of motion under these conditions in Cartesian coordinate system Oxz for the exponential law of change $v = \kappa z^n$ of velocity of the interceptor is given below.

The coordinate $z(\psi)$ of the interceptor is determined for different exponents n from Table 1.13. The coordinate x of the interceptor is found from the formula

$$x = x_0 - \frac{1}{n-2}\int_{\psi_0}^{\psi} z(\xi)\cot^2\xi\,d\xi - \frac{x x_{t0}}{v_t(n-2)} \times \times \int_{\psi_0}^{\psi} z^n(\xi)\frac{\cos^2\xi}{\sin\xi}\,d\xi, \qquad (1.131)$$

where $z(\xi)$ and n are found in Table 1.13, and the integrals are calculated by some approximation method.

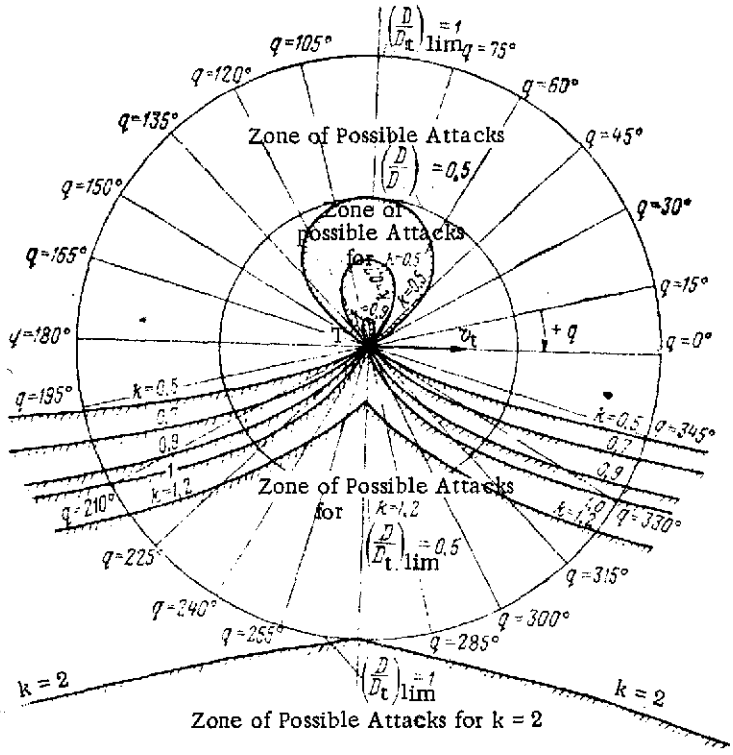

Figure 1.23. Approximate Zone of Attack with Respect to g-Force for Triangulation Method.

TABLE 1.11 *

Velocity Ratio k	1,000	0,909	0,833	0,769	0,714	0,667	0,625	0,587	0,556	0,526	0,500
Initial Angle $\Delta\psi_0^*$	37°6'	28°2'	23°23'	20°5'	17°34'	14°52'	13°49'	12°26'	11°16'	10°16'	9°24'

*Note: Commas indicate decimal points.

TABLE 1.12

Velocity Ratio	Position of Target and Interceptor Relative to Guidance Center	Zone of Attack
$k > 1$	$\left(\frac{D}{D_t}\right)_{lim} > 0$	Entire Guidance Plane
	$\left(\frac{D}{D_t}\right)_{lim} < 0$	
$k \leqslant 1$	$\left(\frac{D}{D_t}\right)_{lim} > 0$	Part of the Guidance Plane Bounded by a Curve (1.130)
	$\left(\frac{D}{D_t}\right)_{lim} < 0$	

TABLE 1.13

Exponent n	Law of Change of Velocity $v = xz^n$	Coordinate $z(\psi)$
$\frac{3}{2}$	$v = xz^{\frac{3}{2}}$	$\left(\frac{\sin\psi}{\sin\psi_0}\right)^2\left[\frac{1}{\sqrt{z_0}} - \frac{xz_0(\psi-\psi_0)}{v_t}\sin\psi_0\right]^{-2}$
$\frac{5}{3}$	$v = xz^{\frac{5}{3}}$	$\left(\frac{\sin\psi}{\sin\psi_0}\right)^3\left[z_0^{-\frac{2}{3}} - \frac{2xz_0}{v_t}\sin^2\psi_0(\cos\psi_0 - \cos\psi)\right]^{-\frac{3}{2}}$

TABLE 1.13 (CONTINUED)

Exponent n	Law of Change of Velocity $v = xz^n$	Coordinate $z(\psi)$
$\frac{7}{4}$	$v = xz^{\frac{7}{4}}$	$\left(\frac{\sin\psi}{\sin\psi_0}\right)^4\left[z_0^{-\frac{3}{4}} - \frac{3xz_0}{v_t}\sin^3\psi_0\left(\frac{1}{2}\psi - \frac{1}{2}\psi_0 - \right.\right.$ $\left.\left. - \frac{1}{4}\sin 2\psi + \frac{1}{4}\sin 2\psi_0\right)\right]^{-\frac{4}{3}}$
$\frac{9}{5}$	$v = xz^{\frac{9}{5}}$	$\left(\frac{\sin\psi}{\sin\psi_0}\right)^5\left[z_0^{-\frac{4}{5}} - \frac{4xz_0}{v_t}\sin^4\psi_0(\cos\psi - \cos\psi_0 + \right.$ $\left. + \frac{1}{3}\cos^3\psi - \frac{1}{3}\cos^3\psi_0\right)\Big]^{-\frac{5}{4}}$
$\frac{11}{6}$	$v = xz^{\frac{11}{6}}$	$\left(\frac{\sin\psi}{\sin\psi_0}\right)^6\left[z_0^{-\frac{5}{6}} - \frac{5xz_0}{v_t}\sin^5\psi_0\left(\frac{3}{8}\psi - \right.\right.$ $- \frac{3}{8}\psi_0 - \frac{1}{4}\sin 2\psi + \frac{1}{4}\sin 2\psi_0 +$ $\left.\left. + \frac{1}{32}\sin 4\psi - \frac{1}{32}\sin 4\psi_0\right)\right]^{-\frac{5}{6}}$
1	$v = xz$	$\frac{\sin\psi}{\sin\psi_0}\left(\frac{\tan\frac{\psi}{2}}{\tan\frac{\psi_0}{2}}\right)^{\frac{xz_0}{v_t}} z_0$
3	$v = xz^3$	$\frac{\sin\psi_0}{\sin\psi}\left[z_0^{-2} + \frac{xz_0}{v_t\sin^2\psi_0}\left(\frac{\cos\psi_0}{\sin^2\psi_0} - \frac{\cos\psi}{\sin^2\psi} + \right.\right.$ $\left.\left. + \ln\frac{\tan\frac{\psi}{2}}{\tan\frac{\psi_0}{2}}\right)\right]^{-\frac{1}{2}}$

Note: Here x = const is coefficient of proportionality between the coordinate and velocity.

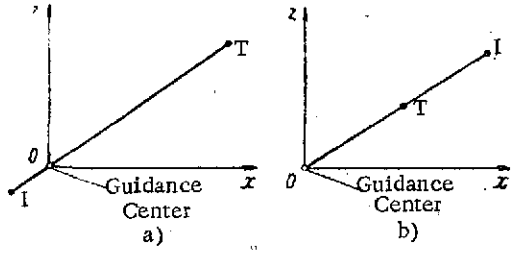

Figure 1.24. Possible Locations of the Interceptor and Target Relative to the Guidance Center.

1.6. Proportional Closing Method

The guidance method in which the angular velocity of the interceptor velocity vector is proportional to the angular velocity of the "interceptor--target" line is called the *method of proportional closing*. The equations for the method are given in Table 1.14.

TABLE 1.14

Coordinate System	Equations of Motion	Equations of Guidance Method
Cartesian	$\dot{x}_t = v_t \cos \psi_t$	$\dot{\psi} = p\omega_l$
	$\dot{z}_t = v_t \sin \psi_t$	
	$\dot{x} = v \cos \psi$	
	$\dot{z} = v \sin \psi$	
Polar	$\dot{D} = -v \cos \phi - v_t \cos q$	$\dot{\psi} = p\omega_l$
	$q = \frac{v_t \sin q - v \sin \phi}{D} + \omega_t$	
	$\omega_l = \frac{v_t \sin q - v \sin \phi}{D}$	p is a constant proportionality coefficient

1.6.1. Equations of Motion and Proportional Closing Method

Let the target fly arbitrarily at velocity v_t and turn at angular velocity ω_t, whose positive direction is shown in Figure 1.25.

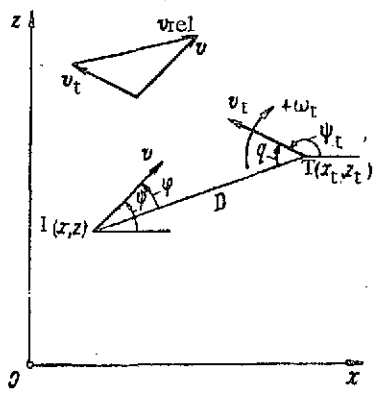

Figure 1.25. Elements of the Kinematic Trajectory for the Method of Porportional Convergence.

The positions of the interceptor and target in Cartesian coordinate system Oxz, whose origin is arbitrary point O, and whose Ox and Oz are oriented arbitrarily, are defined by the coordinates x, z and x_t, z_t, respectively. The position of the interceptor relative to the target in polar coordinate system TDq, whose origin is connected to the target, is determined by the distance D between the target and interceptor and target heading q, which gives the angular position of the interceptor relative to the target velocity vector. /62

The orientation of the velocity vectors of the interceptor and target in Cartesian coordinate system Oxz is given by angles ψ and ψ_t between the Ox axis and the corresponding vector v or v_t. The orientation of the velocity vector of the interceptor in the polar coordinate system is given by target heading ϕ between the "interceptor-target" line and velocity vector v_t.

The equations of motion of the interceptor and target and the flight conditions of the interceptor for the method of proportional closing written in projections onto the axes of the Cartesian and polar coordinate systems are given in Table 1.14.

Initial conditions. For t = 0 the coordinates are given:

— in the Cartesian system

$$x_t(0) = x_{t0}; \qquad x(0) = x_0; \qquad \psi(0) = \psi_0;$$

$$z_t(0) = z_{t0}; \qquad z(0) = z_0;$$

— in the polar system

$$D(0) = D_0; \qquad q(0) = q_0.$$

By adding to these conditions the laws for change of target velocity $v_t(t)$, interceptor velocity $v(t)$, target course $v_t(t)$, or the angular velocity of the target velocity vector $\omega_t(t)$, we obtain a closed system of differential equations in Cartesian and polar coordinates.

1.6.2. Solution of Equations of Motion in Polar Coordinate System for Interception by Method of Proportional Closing of a Rectilinearly and Uniformly Flying Target ($\omega_t = 0$, $v_t = \text{const}$) with Constant Interceptor Velocity ($v = \text{const}$) /63

Let the target fly in a straight line ($\omega_t = 0$; $\Psi_t = \text{const}$) uniformly ($v_t = \text{const}$). The interceptor also flies uniformly ($v = \text{const}$). The solution of the equations of motion for certain values of constant p under these conditions is given below.

1. The solution of equations of motion for p = 2 is given by the following formulae.

The distance between the interceptor and target is

$$D = D_0 \left[\frac{\sin\varphi + k\sin(\varphi - \varepsilon_1)}{\sin\varphi_0 + k\sin(\varphi_0 - \varepsilon_1)} \right]^{\frac{1-k^2}{k^2+2k\cos\varepsilon_1+1}} \times \\ \times \exp\left[-\frac{2k(\varphi - \varphi_0)\sin\varepsilon_1}{k^2 + 2k\cos\varepsilon_1 + 1} \right], \tag{1.132}$$

where

$$\varepsilon_1 = \psi_0 - 2\phi_0. \tag{1.133}$$

The angular velocity of the "interceptor-target" line:

$$\omega_l = \dot{q} = -\dot{\varphi} = -\frac{v_t}{kD_0} \left[\sin\varphi_0 - k\sin(\varepsilon_1 - \varphi_0) \right] \left(\frac{D}{D_0} \right)^{\frac{2k(k+\cos\varepsilon_1)}{1-k^2}} \times \\ \times \exp\left[2k \frac{(\varphi - \varphi_0)\sin\varepsilon_1}{1-k^2} \right]. \tag{1.134}$$

2. The solution of equations of motion for p = 3 is given by the following formulae.

The distance between the interceptor and target is

$$D = D_0 \left[\frac{k\sin q - \sin 2(q + \varepsilon_1)}{k\sin q_0 - \sin 2(q_0 + \varepsilon_1)} \right]^{\frac{1}{2}} \times \\ \times \prod_{i=1}^{4} \left| \frac{\cot\frac{1}{2}(q_i + \varepsilon_1) - \cot\frac{1}{2}(q + \varepsilon_1)}{\cot\frac{1}{2}(q_{0i} + \varepsilon_1) - \cot\frac{1}{2}(q_0 + \varepsilon_1)} \right|^{-3B_i\cot\varepsilon_1}, \tag{1.135}$$

where

$$\varepsilon_1 = \frac{3(\pi - q_0) + \psi_0}{2}; \tag{1.136}$$

$$B_i = -\frac{\cos q_i \operatorname{cosec}^2 \frac{1}{2}(q_i + \varepsilon_1)}{2[2\cos 2(q_i + \varepsilon_1) - k\cos(q_i + \varepsilon_1)]^2}; \tag{1.137}$$

q_i are the roots of the equation $k \sin q + \sin 2(q + \varepsilon_1) = 0$. (1.138)

Special case. Let $\varepsilon_1 = 0$. Then the distance between the interceptor and the target is determined from formulae /64

$$D = D_0 \left| \frac{1 + \cos q}{1 + \cos q_0} \right|^{\frac{1-k}{2(k+2)}} \cdot \left| \frac{1 - \cos q}{1 - \cos q_0} \right|^{\frac{1+k}{2(2-k)}} \times \times \left| \frac{k - 2\cos q}{k - 2\cos q_0} \right|^{\frac{2(1-k^2)}{4-k^2}}. \tag{1.139}$$

3. The solution of equations of motion for p = 4 is given by the following formulae.

The distance between the interceptor and target is

$$D = D_0 \left[\frac{\sin 3(q + \varepsilon_1) + k \sin q}{\sin 3(q_0 + \varepsilon_1) + k \sin q_0} \right]^{\frac{1}{8}} \times \times \prod_{i=1}^{3} \left| \frac{\tan(q + \varepsilon_1) - \tan(q_0 + \varepsilon_1)}{\tan(q_i + \varepsilon_1) - \tan(q_0 + \varepsilon_1)} \right|^{-\frac{4M_i k \sin \varepsilon_1}{3(1 - k\cos\varepsilon_1)}}, \tag{1.140}$$

where

$$M_i = \tan\frac{\varepsilon_1}{2} \frac{\cos q_i \sec^2(q_i + \varepsilon_1)}{3\sin(q_i + \varepsilon_1) + k\cos q_i}; \tag{1.141}$$

q_i are the roots of the equation $k \sin q - \cos 3(q + \varepsilon_1) = 0$.

1.7. Direct Guidance Method

The guidance method in which interceptor velocity vector is aimed at a certain point M (Figure 1.26) selected in a definite manner, is called the *direct guidance method.* In particular, this point may be selected from the

condition that determines the position of the interceptor at a certain moment of time at distance l relative to the target. Equations of the guidance method are given in Table 1.15.

1.7.1. Plotting the Kinematic Trajectory for the Direct Guidance Method /65

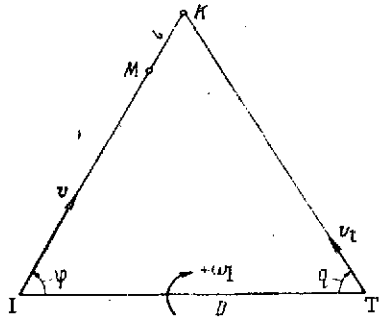

Figure 1.26. Elements of the Kinematic Trajectory for the Direct Closing Method.

Let the target fly uniformly at velocity v_t. For the initial moment of time we are given the distance from the target to the interceptor D_0, the target heading q_0, the distance l and the time t_M.

The trajectory of the target is divided into segments $T_0T_1 = T_1T_2 = \ldots = {}_t\Delta t$, where Δt is the selected time interval (Figure 1.27). The straight line on which segment $T_0P_0 = D_0$ is plotted is drawn through point T_0 at angle q_0 to segment T_0T_1. From the equation

$$\phi_0 = \frac{1}{k\left(1 + \frac{1}{vt_M}\right)} \sin q_0$$

the target bearing ϕ_0 is calculated at the initial moment of time. Then through point I_0 at angle ϕ_0 a straight line is drawn, on which segment I_0I_1, equal to $v\Delta t$, is plotted. Points I_1 and T_1 are connected. Target heading q_1 is measured at point T_1. The target bearing ϕ_1 at point I_1 is calculated for this angle q_1 from the above mentioned formulae. Then a straight line is drawn through point I_1 at angle ϕ_1 to segment I_1T_1. Segment $I_1I_2 = v\Delta t$ is plotted on it. Points I_2T_2 are connected. Then a similar /66 construction is made at points I_2, I_3, ...

The broken line which passes through points I_0, I_1, I_2, ..., when $\Delta t \to 0$ gives the curve for the direct guidance method.

1.7.2. Equations of Motion and the Method of Direct Guidance

Let the target fly rectilinearly and uniformly at velocity v_t (see Figure 1.26). The position of the interceptor relative to the target in polar coordinate system TDq, whose origin is connected to the target, is determined by the distance D between the interceptor and the target and the target heading q, which determines the angular position of the interceptor relative to the target velocity vector.

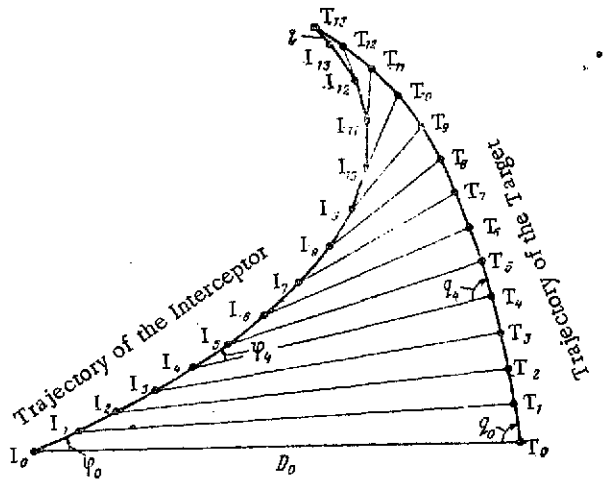

Figure 1.27. Plotting Kinematic Trajectory for Direct Guidance

$$\sin\varphi = \frac{l}{k\left(1+\frac{l}{vt_M}\right)}\sin q,$$

The orientation of the velocity vector of the interceptor is given by the target bearing ϕ between the "interceptor-target" line and the velocity vector v. Orientation of the velocity vectors of the interceptor and target in a Cartesian coordinate system Oxz is given by angles v and v_t between the Ox axis and the corresponding vector v or v_t.

Point M is selected such that at moment t_M when the target reaches point K, which lies on the extension of the interceptor path, the interceptor will be located at point M, i.e., the distance between interceptor and the target at this moment will be l.

The equations of motion of the guidance method which are written in projections on the axes of the Cartesian and polar coordinate systems are given in Table 1.15.

TABLE 1.15

Coordinate System	Equations of Motion	Equations of Guidance Method
Cartesian	$\dot{x}_t = v_t \cos\psi_t$ $\dot{z}_t = v_t \sin\psi_t$ $\dot{x} = v\cos\psi$ $\dot{z} = v\sin\psi$	$\frac{\dot{x}-\dot{x}_t}{\dot{z}-\dot{z}_t} = \frac{x_0 - x_{t0} - v\cos\psi}{z_0 - z_{t0} - v\sin\psi}$
Polar	$\dot{D} = -v\cos\phi - v_t\cos q$ $q = \frac{v_t \sin q - v\sin\phi}{D}$	$\sin\phi = \frac{1}{k(1+\frac{v}{vt_M})}\sin q$

1.7.3. Solution of Equations of Motion in the Polar Coordinate System for Interception by Method of Direct Closing of a Rectilinearly and Uniformly Flying Target ($\omega_t = 0$, $v_t = \text{const}$), for Constant Interceptor Velocity ($v = \text{const}$) and l/t_M /67

Let the target fly rectilinearly and uniformly at velocity v_t. The interceptor also flies uniformly at velocity v. The value l/t_M is considered a constant. The solution of equations of motion under these conditions is found as follows.

The distance between the interceptor and target is

$$D = D_0 \left(\frac{\sin q}{\sin q_0}\right)^{\frac{k}{\lambda_1 - k}} \left(\frac{\sqrt{1-\lambda_1^2 \sin^2 q_0} + \lambda_1 \cos q_0}{\sqrt{1-\lambda_1^2 \sin^2 q} + \lambda_1 \cos q}\right)^{\frac{\lambda_1}{\lambda_1 - k}} \times$$

$$\times \left[\frac{\left(\sqrt{1-\lambda_1^2 \sin^2 q} + \cos q\right)\left(\sqrt{1-\lambda_1^2 \sin^2 q_0} - \cos q_0\right)}{\left(\sqrt{1-\lambda_1^2 \sin^2 q} - \cos q\right)\left(\sqrt{1-\lambda_1^2 \sin^2 q_0} + \cos q_0\right)}\right]^{\frac{1}{2(\lambda_1 - k)}}. \quad (1.142)$$

The flight time of the interceptor on the kinematic trajectory from point (D_0, q_0) to point (D, q) is

$$t = \frac{1}{v_t (1 - k\lambda_1)} \int_{q_0}^{q} \frac{D(q)}{\sin q} dq. \quad (1.143)$$

Target bearing is

$$\varphi = \arcsin(\lambda_1 \sin q), \quad (1.144)$$

where

$$\lambda_1 = \frac{v_t t_M}{v_t t_M + l}. \quad (1.145)$$

CHAPTER 2

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AIRCRAFT AS A CONTROLLED OBJECT

SYMBOLS

$A(\omega)$ -- Amplitude-frequency response.
a_{ij} -- Dynamic coefficients of aircraft in longitudinal motion.
b_{ij} -- Dynamic coefficients of aircraft in lateral motion.
f_a -- Natural frequency of aircraft vibrations.
G -- Aircraft weight.
g -- Gravity acceleration.
h -- Modulus stability margin, a transient (temporal) function.
j -- Imaginary unit.
K -- Transfer coefficient; amplification factor.
$M(s)$ -- Characteristic polynomial.
M_x -- Roll moment.
M_y -- Yaw moment.
M_z -- Pitch moment.
m -- Aircraft mass.
m_x -- Roll moment factor.
m_y -- Yaw moment factor.
m_z -- Pitch moment factor.
P -- Thrust.
$P(\omega)$ -- Real frequency response.
$Q(\omega)$ -- Imaginary frequency response.
s -- A complex value; independent variable of transforms.
T -- Time constant.
T_a -- Period of natural vibrations of aircraft.
T_{con} -- Time of the transient process (control time).
t -- Time.
V -- Airspeed of center of mass of aircraft; Lyapunov's function.
X -- Drag.

$x_{in}(t)$ -- Input variable (original).
$x_{in}(s)$ -- Transform of input variable.
$x_{out}(t)$ -- Output variable (original).
$x_{out}(t)$ -- Transform of output variable.
Δx (t) -- Deviation (perturbation).
Y -- Aircraft lift.
Y (s) -- Transfer function.
Y ($j\omega$) -- Amplitude-phase frequency response. /69
Z -- Lateral force.
α -- Angle of attack.
β -- Angle of skid.
γ -- Angle of roll; phase stability margin.
Δ -- Increment (deviation) of any value.
δ -- Deflection of aircraft control elements.
δ (t) -- Delta function.
δ_h -- Angle of deflection of pitch control elements (elevators).
δ_d -- Angle of deflection of yaw control elements (rudders).
δ_a -- Angle of deflection of roll control elements (ailerons).
δ_{eng} -- Angle of deflection of turning engine.
ε -- Control error (misalignment).
ε (∞) -- Steady state (static) error.
ε_{dyn} -- Dynamic error.
θ -- Angle of inclination of trajectory to horizon.
ϑ -- Pitch angle.
ξ -- Relative attenuation (damping) coefficient.
σ -- Overcontrol.
τ -- Time constant.
φ (ω) -- Phase-frequency characteristic.
ψ_C -- Velocity course angle.
ψ -- Aircraft course angle.
ω -- Frequency of oscillations.
ω_a -- Natural frequency of oscillations.
ω_{free} -- Frequency of free oscillations.
ω_{cut} -- Cut off frequency.
ω_x ω_y ω_z -- Projections of angular velocity vector onto connected axes.
[1] -- Single stage action.

This chapter contains a description of the dynamic properties of aircraft as a component of the automatic flight control system during determined actions. The aircraft is viewed as one of several components of the control system, namely the controlled object. During investigation of the dynamic

properties of aircraft in Sections 2.2 and 2.3 it is suggested that the perturbed motion of aircraft be described by a system of linear differential equations with constant coefficients. Aircraft properties are studied by methods of the automatic control theory which are used both for studying the entire control system as a whole as well as its separate elements and components. Basic facts and methods of linear automatic control theory are illustrated preliminarily in Section 2.1. Section 2.4 introduces certain methods of general mechanics which are suitable for analytical construction and analysis of aircraft motions in circumstances when these motions are described by a system of ordinary nonlinear differential equations.

In cases, when nonlinear problems cannot be analyzed by methods described in this chapter, one should resort to numerical methods of solution (see Chapter 4). /70

2.1. Basic Information From Linear Automatic Control Theory

2.1.1. Aircraft Motion Control System

Aircraft motion is controlled if there is control or regulation of the parameters of its motion in space. The most common method of purposeful change of motion parameters is the creation and use of controlling forces and moments, called *controlling actions*.

In addition to the controlling actions perturbing forces and moments also act on the aircraft in flight and are called *perturbing actions*.

All variable equation aircraft motion systems (see Sections 2.2.1, 2.3.1) are subdivided into input $x_{in}(t)$ and output $x_{out}(t)$ systems. Their definitions are given in Table 2.1.

Effective creation and use of controlling actions requires a control apparatus, the purposes of which are:

--collection of the necessary information about the actual motion parameters;

--formation of a control signal based on the above information;

--actuation of aircraft control elements.

A dynamic system consisting of the aircraft and the control apparatus is a *control system*.

The majority of aircraft automatic control systems are *closed systems*, i.e., systems with *feedback* (Table 2.1). The operation of such systems is based on comparison of actual and required motion parameters and on minimization of the resulting difference to zero. /72

Functional and structural diagrams (see Table 2.1) are methods for depicting control systems. Structural diagrams are ordinarily represented as a set of the simplest, i.e., elementary standard components, whose characteristics are given in Table 2.3. Because a high-order differential equation can be replaced by the corresponding number of lower-order equations several elementary components of a structural diagram may correspond to one element of a functional diagram.

Representation of a control system as a set of elements of one or another diagram is ordinarily done in the assumption that the state of each element is described by only two specific variables--the input $x_{in}(t)$ and the output $x_{out}(t)$. If either of these values is the algebraic sum of several other variables, an adder is included in the diagram, which performs the indicated summation. Conversion of the structural diagrams is discussed in Section 2.1.6.

Examples of control systems, namely different types of functional diagrams of aircraft angular motion control systems, are shown in Figures 2.1, 2.2, and 2.3.

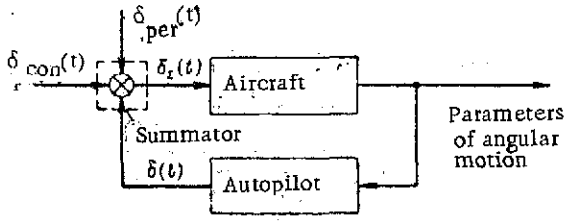

Figure 2.1. Example of Angular Motion Control System. $\delta_{per}(t)$ is Deflection of Control Element, Equivalent to Autopilot Signal; $\delta_{con}(t)$ is the Deflection of Control Element, Equivalent to Controlling Action; $\delta_{per}(t)$ is Deflection of Control Element, Equivalent to Perturbation; $\delta_\Sigma(t)$ is Total Deflection of Control Element.

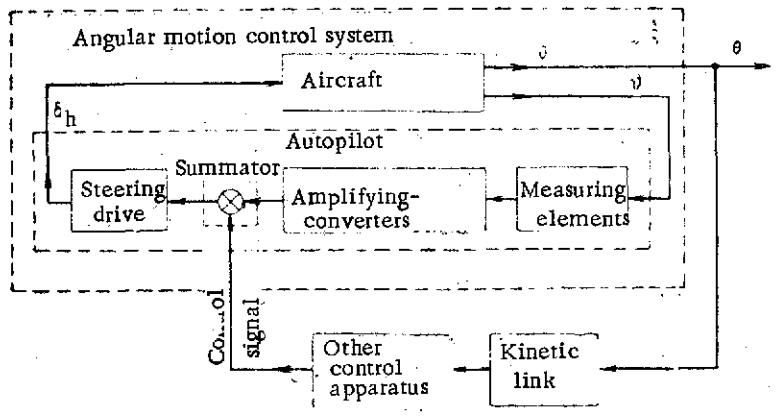

Figure 2.2. Example of Functional Diagram of Control System for Angle of Inclination θ of Trajectory.

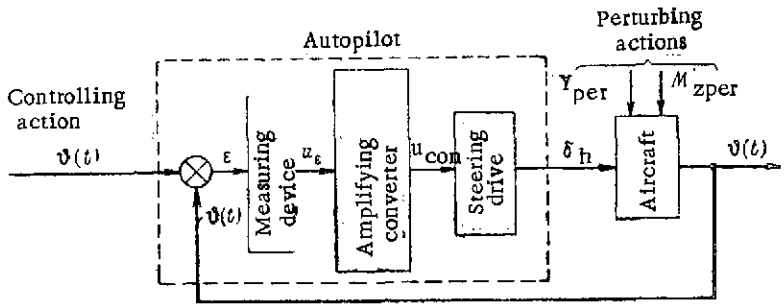

Figure 2.3. Functional Diagram of Automatic Control of Pitch Angle ϑ: $\vartheta(t)$ is Actual (Instantaneous) Value of the Pitch Angle; $\overline{\vartheta}(t)$ is Required (Given) Pitch Angle; ε is Control Error $\varepsilon = \overline{\vartheta}(t) - \vartheta(t)$; u_ε is Error Signal; u_{con} is Control Signal; δ_a is the Angle of Deflection of Elevator; Y_{per} and M_{zper} are the Perturbing Forces and Moments, Respectively.

Translator's Note: Figure 2.4 is not presented in the original Russian text.

2.1.2. Basic Concepts and Definitions of Automatic Control Theory

TABLE 2.1

Number	Concept	Symbol	Definition	Example	Remark
1	Control action	$g(t)$	Action on the aircraft necessary for creating controlling forces and moments	Deflection of control elements	Control action determines required aircraft motion parameters.
2	Perturbation	$f(t)$	All actions on the aircraft that deflect the motion parameters from their required values	Wind actions on aircraft, random deviations of control elements etc.	As a result of continuous action of perturbations the actual output variables of the aircraft always differ from required
3	Input variable	$x_{in}(t)$	A variable whose change in time is equivalent to the nature of the controlling or perturbing action.	δ_h, δ_d, δ_a, etc.	

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TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
4	Output variable	$x_{out}(t)$	Variable whose change in time determines the aircraft's reaction to control action or perturbation	ϑ, α, $\dot{\theta}$, V, H etc.	
5	Aircraft motion control system	CS	Dynamic system consisting of aircraft as controlled object and the control apparatus	Figures 2.1, 2.2, 2.3.	CS which stabilizes angular motion parameters in presence of perturbations is called *stabilization system.*
6	Autopilot	AP	Control apparatus for regulating or stabilizing angular motion parameters	Figures 2.1, 2.2, 2.3.	

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
7	Feedback (*feedback signal*)	$x_{fb}(t)$	Part (circuit) of control system which closes the system and is intended for producing signal $x_{fb}(t)$ based on information on actual motion parameters	Figures 2.1, 2.2, 2.3.	Feedback signal $x_{fb}(t)$ is output variable for feedback circuit.
8	Positive feedback	—	Type of feedback in which signal $x_{fb}(t)$ is added to main input signal (with input variable).	$x_{in}(t) + x_{fb}(t)$	Positive feedback impairs dynamic properties of CS and is only used for increasing transfer coefficient of closed system.
9	Negative feedback		Type of feedback in which signal $x_{fb}(t)$ is subtracted from main signal (from input variable)	$x_{in}(t) - x_{fb}(t)$	Negative feedback improves dynamic properties of CS and therefore is often called *stabilizing feedback* (here the transfer coefficient of the closed system is decreased)

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
10	Open control system		Control system without feedback circuit, i.e., does not use information about actual variable used as output variable for the entire system		Part of the closed system obtained in the absence (break) of feedback circuit is also an open system.
11	Closed control system		Control system which uses feedback signal, i.e., information about actual output variable	Figures 2.1, 2.2, 2.3.	A closed system usually operates on principle of minimization of difference of the type $x_{in}(t) - x_{fb}(t)$
12	Functional diagram (block diagram)		Method of depicting dynamic systems, a set of interconnected elements distinguished by type of function and purpose in a given system.	Figures 2.1, 2.2, 2.3.	Separation of elements in functional system often coincides with their structural separation.

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
13	Structural diagram		Method of depicting dynamic systems, the set of interconnected elements distinguished by type of dynamic properties, i.e., by type of motion equations corresponding to them	Figures 2.9, 2.10, 2.11.	The transfer function is the equivalent of dynamic properties of element in structural system; elements of structural systems are called *components.*
14	Basic component		Element of structural diagram, whose motion equation exponent is not higher than two and cannot be reduced without loss of reality of the coefficients.	See Table 2.3.	
15	Typical basic component		Any of group of basic components which are related by common dynamic properties, i.e., by form of motion equation.	See Table 2.3.	

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
16	Single jump or a single step action	$x_{in} = [1]$	Typical action of form $x_{in} = \begin{matrix} 1 \text{ when } t \geqslant 0 \\ 0 \text{ when } t < 0 \end{matrix}$	Figure 2.4.	Control action of form $x_{in} = [1]$ characterizes the abrupt change of stable flight conditions, sudden release of load etc. Perturbation of form $x_{in} = [1]$ characterizes the abrupt and continuous perturbations, e.g., abrupt and prolonged gusts of wind etc.
17	Harmonic action	$x_{in} = A_{in} \cos \omega t$	Typical action obeying harmonic law		The given action is characterized by constant frequency ω and certain amplitude A_{in}.

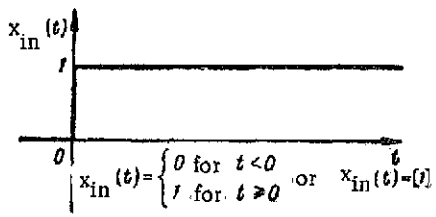


Figure 2.4. Graph of a Single Step Function.

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
18	Pulse action (delta-function type action)	$x_{in} = \delta(t)$	Typical action of form $\delta(t) = \begin{cases} 0 & \text{for } t \neq 0; \\ \infty & \text{for } t = 0; \end{cases}$ $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$		Actually a pulse action is a pulse of arbitrary shape with specific area and brief duration in comparison with expected time of transient process (e.g., abrupt, short gusts of wind, shocks, etc.). /77
19	Static (steady state) condition of component or system		State in which output variable investigated does not change in time. For input actions of the type g(t) = const, f(t) = = const when $t \to \infty$ the state when $t \to \infty$ is considered as steady state.		Properties of component or system in the static condition are called *static properties*. Algebraic relations of the type $x_{out} = Kx_{in}$ are normally equations of statics.

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
20	Transfer coefficient (coefficient of amplification)	K	Ratio of output variable to the input variable in steady state		For given transfer function Y(s) the transfer coefficient K is determined by the relation $K = \lim_{s \to 0} Y(s)$.
21	Dynamic state of the component or system		State in which investigated output variable changes in time.		Properties of component or system in the dynamic state are called *dynamic properties*. Differential equations are usually the equations of dynamics
22	Transient process in terms of variable x_{out}		Process of change of $x_{out}(t)$ from old steady state value to the new steady state value, caused by input (control or perturbing) action.		

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
23	Transient (time) function	$h(t)$	Function $x_{out}(t)$ when $x_{in} = [1]$ and $x_{out}(0) = 0$	See Table 2.3.	
24	Required (given) output variable	$\overline{x}_{out}(t)$	Output variable determined by control action.		When $\varepsilon(\infty) = 0$ we have $\overline{x}_{out}(\infty) = x_{out}(\infty)$
25	Unperturbed output variable (unperturbed motion)	$x^{(0)}_{out}(t)$	Output variable in absence of perturbations		If control actions are also absent then $x^{(0)}_{out}(t) = x_{out}(0)$
26	Steady state output variable	$x_{out}(\infty)$	Output variable in steady state		See remarks on items 19 and 24, Table 2.1.

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
27	Deviation of output variable	$\Delta x(t)$	Difference between actual (instantaneous) output variable and initial value, i.e., $\Delta x(t) = x_{out}(t) - x_{out}(0)$		
28	Perturbation output variable (perturbed motion)	$\Delta x(t)$	Difference between actual (instantaneous output variable and its unperturbed value, i.e., $\Delta x(t) = x_{out}(t) - x^{(0)}_{out}(t)$		If only perturbation is considered then the concept of deviation and perturbation coincide because $x^{(0)}_{out}(t) = x_{out}(0)$
29	Control error	$\varepsilon(t)$	Difference between required (given) output variable and its actual (instantaneous) value, i.e., $\varepsilon(t) = \bar{x}_{out}(t) - x_{out}(t)$		If only perturbation is considered, $\varepsilon(t) = -\Delta x(t)$

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
30	Steady state (static) error	$\varepsilon(\infty)$	Control error in steady state		See remarks for item 19, Table 2.1.
31	Dynamic error	ε_{dyn}	Control error in dynamic state and in absence of perturbations		The cause of dynamic error is aircraft's inertia.
32	Transfer function	Y(s)	Ratio of transform of output variable $x_{out}(s)$ to transform of input variable $x_{in}(s)$ for zero initial conditions	See Table 2.3.	For different pairs of output and input variables the transfer functions are different in general case.

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
33	Frequency amplitude--phase response	$Y(j\omega)$	Form of transfer function when $s = j\omega$ $Y(s) \xrightarrow{s=j\omega} Y(j\omega)$		The characteristic $Y(j\omega)$ is constructed in a complex plane P, jQ.
34	Real frequency response	$P(\omega)$	Real part of $Y(j\omega)$		In this case complex expression $Y(j\omega)$ is represented in the form $Y(j\omega) = P(\omega) + jQ(\omega)$
35	Imaginary frequency response	$Q(\omega)$	Coefficient for imaginary part $Y(j\omega)$		See remark for item 34, Table 2.1.
36	Amplitude--frequency response	$A(\omega)$	Dependence on frequency ω of $A = A_{out}/A_{in}$, where A_{out} is the amplitude of oscillations at output, A_{in} is the amplitude of oscillations at input.	See Table 2.3.	In this case complex expression $Y(j\omega)$ is represented in the form $Y(j\omega) = A(\omega)e^{j\varphi(\omega)}$

TABLE 2.1 (CONTINUED)

Number	Concept	Symbol	Definition	Example	Remark
37	Phase-frequency response	$\varphi(\omega)$	Dependence on frequency ω of phase shift φ of output oscillations relative to the input oscillations.	See Table 2.3.	See remark for item 36, Table 2.1.

2.1.3. Aircraft as Component of Control System

An aircraft as a component of a control system is the controlled object. /81
Its properties are completely described by definitions and concepts of automatic control theory (Table 2.1).

Deflections of the control elements are generally the controlling actions for aircraft, but the input variable is a value which characterizes the position or state of the control elements. In a particular case the angles of deflection of control surfaces δ_h, δ_d, δ_a, the angle of rotation of the wing or engine, change in wing geometry, change in engine thrust, etc. may be the controlling actions of aircraft.

The perturbing action for aircraft may be either the introduction of perturbing forces and moments in the equations of motion, or the corresponding change of the input or output variables, for example:

$M_{z\ per} = M_z^{\delta} \cdot \delta_{per}$, whence $\delta per = M_{z\ per}/M_z^{\delta}$ (see Section 2.2).

Any value which describes aircraft motion, for example α, ϑ, $\dot{\theta}$, H, V, etc., may be the output variable for aircraft.

The choice of output variable is determined by the problem at hand.

The time change of output variables which describe aircraft motion is determined by two factors:

1) Change of input variables, i.e., nature of deflection of control elements;

2) Dynamic properties of aircraft as a component of the control system.

In order to evaluate the dynamic properties of different aircraft one must assume that input actions are identical. For this purpose different types of *standard actions* are widely used. Some of the more widely accepted standard actions are given in Table 2.1.

The dynamic properties of an aircraft as a component of a control system are characterized by *stability* of its motion (see Section 2.1.7) and *by the quality* of transient processes. The values which characterize the quality of aircraft reaction to a sudden action, i.e., which govern the graph of the *transient (temporal) function* h(t) (Table 2.1), are called *transient process quality criteria*. The basic quality criteria of aircraft in terms of the variable x_{out} are indicated in Table 2.2 and Figure 2.5.

The damping properties of aircraft are characterized by the magnitude of overcontrol σ, period T_{free} or frequency ω_{free} of free oscillations.

The speed of response of the aircraft, that is the characteristic of delayed reaction of the system to abrupt action in time, is determined by the time of the transient process T_{con}, by the time to reach the steady state t_0, period T_{free} or frequency ω_{free} of the free oscillations, and also by the integral estimates. /83

Static and Astatic Properties of Aircraft. An aircraft is called *static in relation to perturbation* f(t), if during action approaching a constant value for $t \to \infty$, the deviation of the output variable $\Delta x(t)$ also approaches a constant value depending on the magnitude of action (Figure 2.6, Curve 1).

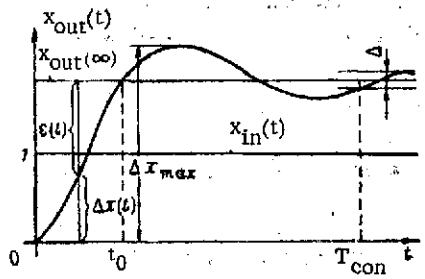


Figure 2.5. Graph of the Stable Transient Process and Main Quality Criteria; $x(\infty) = K$; $x_{in} = [1]$.

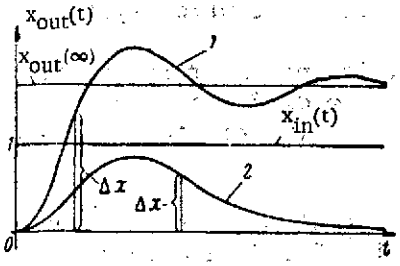


Figure 2.6. Examples of Transient Processes for the Variable x_{out} for Static (Curve 1) and Astatic (Curve 2) Aircraft in Relation to the Perturbation $x_{in}(t)$; $x_{in}(t) = [1]$.

An aircraft is called *astatic in relation to the perturbation* f(t), if during action approaching a constant value for $t \to \infty$, the deviation of the output variable $\Delta x(t)$ approaches zero regardless of the magnitude of action (Figure 2.6, Curve 2).

An aircraft is called *static in relation to the controlling action* g(t), if during action approaching a constant value for $t \to \infty$, the control error $\varepsilon(t)$ also approaches a constant value depending on the magnitude of action (Figure 2.7, Curve 1). /84

An aircraft is called *astatic in relation to the controlling action* g(t), if during action approaching a constant value for $t \to \infty$, the control error $\varepsilon(t)$ approaches zero regardless of the magnitude of action (Figure 2.7, Curve 2).

TABLE 2.2. BASIC QUALITY CRITERIA OF THE TRANSIENT PROCESS CAUSED BY ABRUPT ACTION

Quality Criterion	Symbol	Definition	Figure	Remark
Time of transient process (control time)	T_{con}	Time in which the output variable differs from its steady state value by no more than a certain conditionally small value Δ.	Figure 2.5	Normally $\Delta = 0.05\, x_{out}(\infty)$
Maximum deviation (perturbation)	Δx_{max}	Maximum deviation (perturbation) in the transient process.	Figure 2.5	
Overcontrol	σ	$\sigma = \frac{\Delta x_{max} - x_{out}(\infty)}{x_{out}(\infty)} \cdot 100\%$ for $x_{out}(\infty) \neq 0$		See remark on item 19, Table 2.1.
Period of free oscillations	T_{free}	Time interval between two consecutive maxima of the transient function h(t) (see Table 2.1)		
Frequency of free oscillations	ω_{free}	$\omega_{free} = \frac{1}{T_{free}}\ sec^{-1}$		
Time of first arrival at steady state conditions	t_0	See Figure 2.5		

TABLE 2.2 (CONTINUED)

Quality Criterion	Symbol	Definition	Figure	Remark
Vibration		Number of vibrations during the transient process.		Vibration is also determined by the value of the resonance peak A_{res}, equal to max $A(\omega)$ (See Table 2.1, Item 36.
Linear integral evaluation	I_1	$I_1 = \int_0^\infty \frac{x_{out}(\infty) - \Delta x(t)}{x_{out}(\infty)} dt$		
Quadratic integral evaluation	I_2	$I_2 = \int_0^\infty \left[\frac{x_{out}(\infty) - \Delta x(t)}{x_{out}(\infty)}\right]^2 dt$		

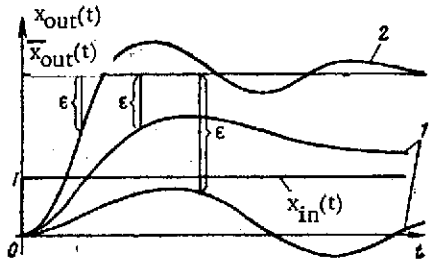


Figure 2.7. Examples of Transient Processes for the Variable x_{out} for Static (Curve 1) and Astatic (Curve 2) Aircraft in Relation to Controlling Action $x_{in}(t)$; $x_{in}(t)$ = [1].

2.1.4. Transfer Function As A Method of Expressing Dynamic Properties of Aircraft

The reaction of aircraft to arbitrary action is characterized by its transfer function. The relation

$$Y(s)=\frac{x_{out}(s)}{x_{in}(s)}=Y_{\frac{x_{out}}{x_{in}}}(s), \qquad (2.1)$$

where $x_{in}(s)$ and $x_{out}(s)$ are the Laplace transforms of the input and output variables, respectively, written for zero initial conditions, is defined as the aircraft *transfer function*.

As is seen, the transfer function Y(s) connects the transforms of two defined variables. Different transfer functions of aircraft, such as the following correspond to different pairs of variables viewed as the input and output (see Section 2.2.4),

$$\frac{\vartheta(s)}{\delta(s)}=Y_{\delta}^{\vartheta}(s)=\frac{K}{T^2s^2+2T\xi s+1}; \quad \begin{matrix} x_{out}=\vartheta; \\ x_{in}=\delta; \end{matrix}$$

$$\frac{\vartheta(s)}{\delta(s)}=Y_{\delta}^{\vartheta}(s)=\frac{K(T_1s+1)}{s(T^2s^2+2T\xi s+1)}; \quad \begin{matrix} x_{out}=\vartheta; \\ x_{in}=\delta. \end{matrix}$$

The transfer functions of certain types of basic components are given in Table 2.3.

It is possible to derive the transfer function Y(s) from the linear differential equation of aircraft motion with constant coefficients of the type (see Sections 2.2.1, 2.3.1 and 2.3.6)

$$a_m x_{out}^{(m)}(t)+a_{m-1}x_{out}^{(m-1)}(t)+\ldots+a_0x_{out}(t)= \\ =b_n x_{in}^{(n)}(t)+b_{n-1}x_{in}^{(n-1)}(t)+\ldots+b_0x_{in}(t) \qquad (2.2)$$

in the following manner.

1) Use the Laplace transform for both parts of the equation (2.2) with consideration of the zero initial conditions

$$M(s)x_{out}(s)=N(s)x_{in}(s), \qquad (2.3)$$

where

$$M(s)=a_ms^m+a_{m-1}s^{m-1}+\ldots+a_0;$$
$$N(s)=b_ns^n+b_{n-1}s^{n-1}+\ldots+b_0;$$

2) Write the transfer function from equation (2.3) as a transform ratio: /89

$$Y(s) = \frac{x_{out}(s)}{x_{in}(s)} = \frac{N(s)}{M(s)}. \qquad (2.4)$$

Example 1. Let the longitudinal perturbed short period aircraft motion be described by the following linear differential equations (see Section 2.2.3) with constant coefficients:

$$\Delta\ddot{\theta} - a_{22}\Delta\dot{\theta} - a_{24}\Delta\alpha = a_{25}\Delta\delta_h;$$
$$\Delta\dot{\theta} - a_{34}\Delta\alpha = 0;$$
$$-\Delta\dot{\theta} + \Delta\theta + \Delta\alpha = 0,$$

which can be reduced to one equation relative to variables $\Delta\delta_h$ and $\Delta\theta$:

$$T^2\Delta\ddot{\theta} + 2T\xi\Delta\dot{\theta} + \Delta\theta = K\Delta\delta_h,$$

where

$$T^2 = -\frac{1}{a_{22}a_{34} + a_{24}};$$
$$2T\xi = \frac{a_{22} - a_{34}}{a_{22}a_{34} + a_{24}};$$
$$K = -\frac{a_{25}a_{34}}{a_{22}a_{34} + a_{24}}.$$

From this equation we derive the transfer function $Y_{\delta}^{\dot{\theta}}(s)$, which is equal to $\dot{\theta}(s)/\delta_h(s)$. By taking the variable $\Delta\delta_h$ for $x_{in}(t)$, and $\Delta\dot{\theta}$ for $x_{out}(t)$, we obtain an equation of motion in the form

$$T^2\ddot{x}_{out}(t) + 2T\xi\,\dot{x}_{out}(t) + x_{out}(t) = K\,x_{in}(t). \qquad (I)$$

By applying the Laplace transform to both parts of equation (I) we obtain (under zero initial conditions)

$$M(s)x_{out}(s) = N(s)x_{in}(s), \qquad (II)$$

where

$$M(s) = T^2s^2 + 2T\xi s + 1; \quad N(s) = K.$$

From equation (II) we write the relation

$$\frac{x_{out}(s)}{x_{in}(s)} = \frac{N(s)}{M(s)} = \frac{\Delta\dot{\theta}(s)}{\Delta\delta_h(s)} = Y_{\delta}^{\dot{\theta}}(s) = \frac{K}{T^2s^2 + 2T\xi s + 1}.$$

The derived expression for $Y_{\delta}^{\dot{\theta}}(s)$ is one of the transfer functions of the aircraft.

The constant coefficients a_m, ..., a_0, b_n, ..., b_0 of equation (2.2), which enter in the expression for Y(s), are called the *parameters* of the transfer function.

The polynomial M(s) is called the *characteristic polynomial*, and the equation

$$M(s)=a_m s^m + a_{m-1}s^{m-1} + \ldots + a_0 = 0 \tag{2.5}$$

is called the *characteristic equation*.

The dynamic properties of aircraft, i.e., reaction in terms of variable /90
x_{out} to arbitrary input action $x_{in}(t)$, are completely determined by the transfer function Y(s) in the following manner.

1. The transform of the input variable $x_{in}(s)$ is determined from the given original of the input variable $x_{in}(t)$ with the direct Laplace transform:

$$x_{in}(s) = L[x_{in}(t)]. \tag{2.6}$$

2. The transform of the output variable is determined from the obtained transform of the input variable $x_{in}(s)$ and the known transfer function of the aircraft Y(s):

$$x_{out}(s) = Y(s)x_{in}(s). \tag{2.7}$$

3. The original of the output variable $x_{out}(t)$ is determined from the obtained transform of the output variable $x_{out}(s)$ using the reverse Laplace transform.

$$x_{out}(t) = L^{-1}[x_{out}(s)]. \tag{2.8}$$

In this manner the transfer function Y(s) permits one to determine the aircraft reaction to any kind of action.

TABLE 2.3: EQUATIONS OF MOTION, TRANSFER FUNCTIONS AND FREQUENCY CHARACTERISTICS OF STANDARD BASIC COMPONENTS

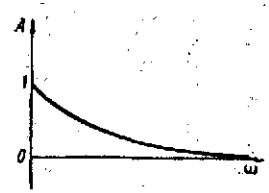
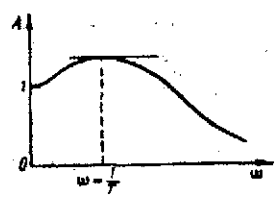
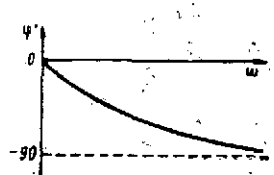
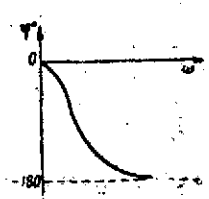
Characteristic	Component	
	Aperiodic Stable Component	Oscillatory Stable Component
Equations of Motion	$T\,\dot{x}_{out}(t) + x_{out}(t) = x_{in}(t)$	$T^2\,\ddot{x}_{out}(t) + 2T\xi\dot{x}_{out}(t) + x_{out}(t) = x_{in}(t);\ 1 - \xi^2 > 0$
Transfer Function	$\frac{1}{Ts+1}$	$\frac{1}{T^2s^2 + 2T\xi s + 1}$
Amplitude-Frequency Response		
Phase-Frequency Response		

TABLE 2.3 (CONTINUED)

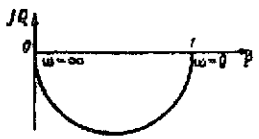
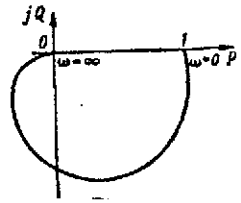
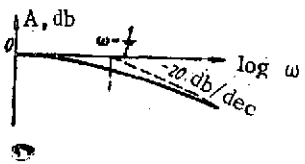
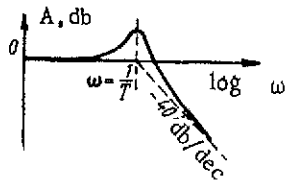
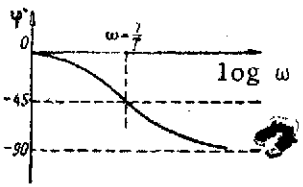
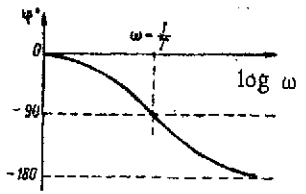
Characteristic	Component	
	Aperiodic Stable Component	Oscillatory Stable Component
Amplitude-Phase Response		
Logarithmic Amplitude-Frequency Response		
Logarithmic Phase-Frequency Response		

TABLE 2.3 (CONTINUED)

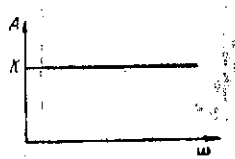
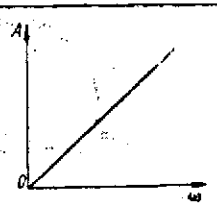
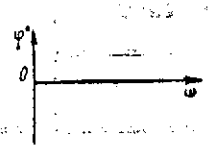
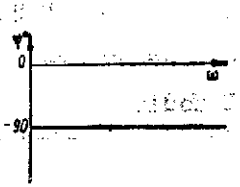
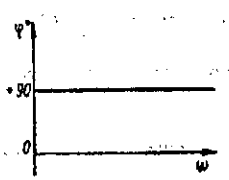
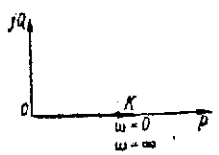
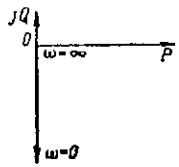
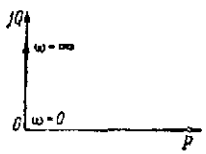
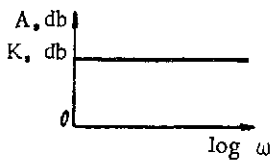
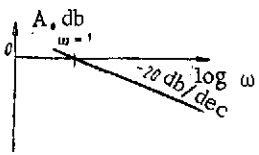
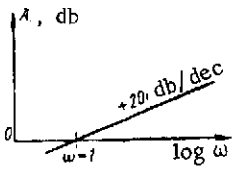
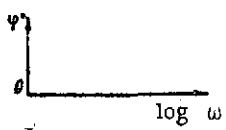
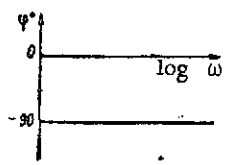
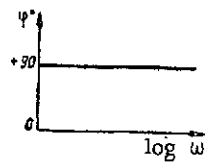
Characteristic	Component		
	Amplifying	Integrating	Differentiating
Equation of Motion	$x_{out}(t) = Kx_{in}(t)$	$x_{out}(t) = \int_0^t x_{in}(t)dt$	$x_{out}(t) = \dot{x}_{in}(t)$
Transfer Function	K	$\frac{1}{s}$	s
Amplitude-Phase Response			
Phase-Frequency Response			

TABLE 2.3 (CONTINUED)

Characteristic	Component		
	Amplifying	Integrating	Differentiating
Amplitude-Phase Response			
Logarithmic Amplitude-Phase Response			
Logarithmic Phase-Frequency Response			

Remarks. 1. T - Time constant
2. ξ - Relative attenuation coefficient
3. When $1 - \xi^2 < 0$ the oscillatory component breaks down into two consecutive aperiodic components.

Basic Properties of Transfer Functions

1. Transfer functions of aircraft are rational functions of the form

$$Y(s)=\frac{N(s)}{M(s)}=\frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_0}, \tag{2.9}$$

where $n \leqslant m$.

2. All parameters of transfer functions $a_0, \ldots, a_m; b_0, \ldots, b_n$ are real.

3. Complex zeros of the transfer function and its complex poles may only be complex-conjugate.

4. The transfer coefficient K for variable x_{out} is determined by the transfer function in the following manner:

$$\lim_{s\to 0} Y(s)=K. \tag{2.10}$$

5. All poles of the transfer function of an aircraft, which is stable with respect to variable x_{out}, have negative real parts.

2.1.5. Frequency Responses as a Means of Expressing Dynamic Properties of Aircraft

The reaction of an aircraft to any kind of input action permits one to determine its transfer function Y(s). Study of the aircraft reaction to harmonic action of the type

$$x_{in}(t) = A_{in}\cos \omega t \tag{2.11}$$

is of great practical interest.

The harmonic of the input variable of an aircraft (i.e., of the control element) at constant frequency ω and with amplitude A_{in} causes a harmonic vibration of the output variable of the type /91

$$x_{out}(t) = A_{out}\cos (\omega t + \varphi). \tag{2.12}$$

with the same frequency ω, but with a changed amplitude A_{out} and with a phase shift φ. Here the variable s in the expression for the transfer function Y(s) acquires a purely imaginary value $s = j\omega$, where ω is the frequency of vibrations fed to the input. The transfer function $Y(j\omega)$ in this case is called the *frequency response*.

The following types of frequency responses are distinguished.

The *amplitude-frequency response* (AFR) represents the dependence on frequency ω of the relation

$$A(\omega) = A_{out}/A_{in}. \tag{2.13}$$

The *phase-frequency response* (PFR) represents the dependence on frequency ω of phase shift $\varphi(\omega)$ of the output vibrations relative to the input vibrations.

The *frequency response* $Y(j\omega)$ combines the properties of the AFR and the PFR. The expression for the frequency characteristic of an aircraft can be obtained by assuming $s = j\omega$ in the expression of the corresponding transfer function:

$$Y(s) \xrightarrow{s=j\omega} Y(j\omega) = \frac{N(j\omega)}{M(j\omega)} = Y_{x_{in}}^{x_{out}}(j\omega). \tag{2.14}$$

The complex expression $Y(j\omega)$ (see Section 2.1.3.) can be represented in exponential form

$$Y(j\omega) = A(\omega)e^{j\varphi(\omega)}, \tag{2.15}$$

where $A(\omega)$ is the AFR, and $\varphi(\omega)$ is the PFR, or in algebraic form

$$Y(j\omega) + P(\omega) + jQ(\omega), \tag{2.16}$$

where $P(\omega)$ is the *real frequency response*; $Q(\omega)$ is the *imaginary frequency response*.

The frequency response $Y(j\omega)$, constructed graphically in complex axes $P(\omega)$ and $jQ(\omega)$ from equation (2.16), is called the *amplitude-phase response* (APR).

A change of sign of ω changes only the phase shift sign, and preserves the ratio of amplitudes unchanged. Consequently, the APR is symmetrical relative to the real axis P and is usually constructed only for frequency ω in the range $0 \leqslant \omega < \infty$.

The APR is related to the AFR and the PFR by the relations

$$\left.\begin{aligned} P(\omega) &= A(\omega)\cos\varphi(\omega); \\ Q(\omega) &= A(\omega)\sin\varphi(\omega). \end{aligned}\right\} \tag{2.17}$$

The AFR and the PFR are related to the APR by the relations /92

$$\left.\begin{aligned} A(\omega) &= \sqrt{P^2(\omega)+Q^2(\omega)}; \\ \tan\varphi(\omega) &= Q(\omega)/P(\omega). \end{aligned}\right\} \qquad (2.18)$$

The relation of different types of frequency responses is shown graphically in Figure 2.8.

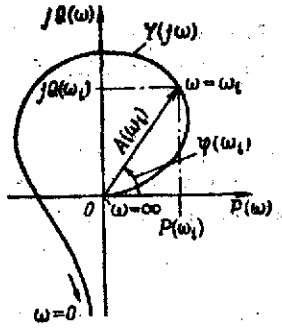


Figure 2.8. Relation of Different Types of Frequency Responses: Y(jω) is the Amplitude Phase Response of Component with Transfer Function Y(s); ω_i is the Fixed Frequency ω

$$Y(j\omega) = A(\omega)\,e^{j\varphi(\omega)} = P(\omega) + jQ(\omega);$$
$$A(\omega_i) = \sqrt{P^2(\omega)+Q^2(\omega)};$$
$$\tan\varphi(\omega_i) = \frac{Q(\omega)}{P(\omega)}$$

The AFR and PFR are also used in logarithmic form.

Relation 20 log A(ω) to log ω is called the *logarithmic amplitude-frequency response* (LAR). Here 20 log A(ω) is considered the value A(ω), expressed in decibels (dB), i.e.,

$$20 \log A(\omega) = A(\omega) \text{ dB}. \qquad (2.19)$$

The relation of phase shift φ(ω), expressed in degrees, to log ω is called the *logarithmic phase-frequency response* (LPR).

The LAR and LPR are constructed one under another with combined scale on the horizontal axis (frequency axis), and the frequency axis is marked off by values frequency ω. The segment of the horizontal scale which corresponds to a ten-fold increase in frequency is called a *decade*, and a segment which corresponds to doubling of frequency, is an *octave*. The slope of parts of the LAR is noted by an increase or decrease of the ordinate throughout one decade, for example, 20 dB/dec (see Table 2.3).

The frequency at which the logarithmic amplitude-frequency response intersects the frequency axis is called the cutoff frequency ω_{cut}.

Examples of different types of frequency responses for a number of standard basic components are given in Table 2.3.

2.1.6. Transformations of Structural Diagrams

If several components of the structural diagram correspond to the aircraft as a component of a functional system, its transfer function and frequency responses as a whole are calculated as a function of the method of connecting the components. Then it is necessary to follow the rules presented in Table 2.4. /95

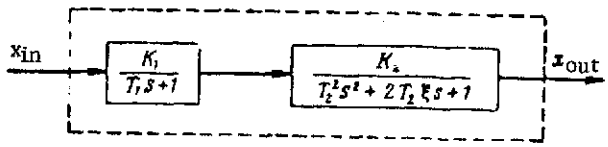


Figure 2.9. Example of Series Connection of Components:

$$Y_1(s)=\frac{K_1}{T_1 s+1};\quad Y_2(s)=\frac{K_2}{T_2^2 s^2+2T_2\xi s+1};$$

$$\frac{x_{out}(s)}{x_{in}(s)}=Y(s)=Y_1(s)\cdot Y_2(s)=\frac{K}{(T_1 s+1)\left(T_2^2 s^2+2T_2\xi s+1\right)};$$

$$K=K_1K_2$$

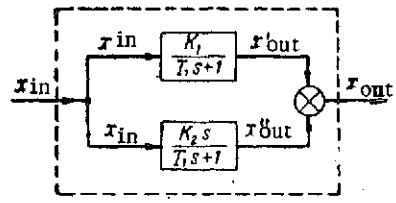


Figure 2.10. Example of Parallel Connection of Components:

$$x_{out}=x'_{out}+x''_{out};$$

$$Y_1(s)=\frac{K_1}{T_1 s+1};\quad Y_2(s)=\frac{K_2 s}{T_1 s+1};$$

$$\frac{x_{out}(s)}{x_{in}(s)}=Y(s)=Y_1(s)+Y_2(s)=K_1\frac{\tau s+1}{T_1 s+1}$$

$$\tau=\frac{K_2}{K_1}$$

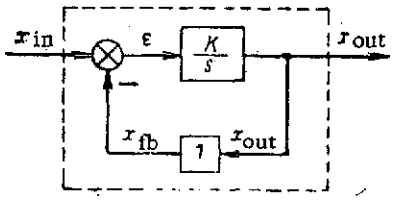


Figure 2.11. Example of Closing Component By a Single Negative Feedback.

$$Y_1(s)=\frac{K}{s};\quad Y_2(s)=1;$$

$$\frac{x_{out}(s)}{x_{in}(s)}=Y(s)=\frac{Y_1(s)}{1+Y_1(s)Y_2(s)}=\frac{1}{Ts+1};$$

$$T=\frac{1}{K}$$

2.1.7. Stability of Motion

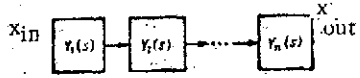
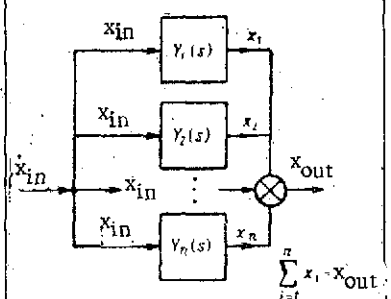
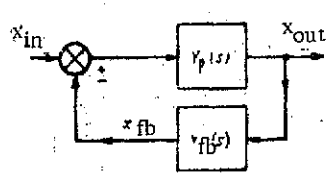
Any aircraft motion which is characterized by the relation $x_{out}(t)$ can be represented in the form

$$x_{out}(t)=x^{(0)}_{out}(t)+\Delta x(t), \qquad (2.20)$$

where $x^{(0)}_{out}(t)$ is called *unperturbed motion*, and $\Delta x(t)$ is *perturbed motion* (see Table 2.1). In the absence of controlling actions the non-zero initial values of $\Delta x(0)$ are the result of external disturbances applied to the aircraft and are called *initial perturbations*.

Aircraft motion stability in terms of variable x_{out} is determined by its capacity to return after some time to unperturbed motion $x^{(0)}_{out}(t)$ after the action of external perturbations ceases. Consequently, motion stability is determined by the behavior of the perturbed motion $\Delta x(t)$ when $t > 0$, and aircraft motion stability is defined as the stability of unperturbed motion in the sense of strength and ability to withstand external disturbances. Later we shall present the definition of stability given by A. M. Lyapunov.

TABLE 2.4. RULES FOR CONVERSION OF STRUCTURAL SYSTEMS

Method of Connecting Components	Structural Diagram	Transfer Function	Frequency Characteristics
Series Connection	 See example in Figure 2.9	$Y(s)=\frac{x_{out}(s)}{x_{in}(s)}=\prod_{l=1}^{n} Y_l(s)$	$A(\omega)=\prod_{l=1}^{n} A_l(\omega)$ $\varphi(\omega)=\sum_{l=1}^{n}\varphi_l(\omega)$
Parallel Connection	 See example in Figure 2.10	$Y(s)=\frac{x_{out}(s)}{x_{in}(s)}=\sum_{l=1}^{n} Y_l(s)$	$P(\omega)=\sum_{l=1}^{n} P_l(\omega)$ $Q(\omega)=\sum_{l=1}^{n} Q_l(\omega)$
Closed System (closing a component with feedback)	 See example in Figure 2.11	$Y(s)=\frac{x_{out}(s)}{x_{in}(s)}=$ $=\frac{Y_p(s)}{1\pm Y_p(s)Y_{fb}(s)}$ "+" For Negative Feedback "-" For Positive Feedback	$Y(j\omega)=\frac{Y_p(j\omega)}{1\pm Y_p(j\omega)Y_{fb}(j\omega)}$

Unperturbed motion $x_{out}^{(0)}(t)$ is called *stable* in relation to Δx, if another positive integer $\delta(\varepsilon)$ is found for any small positive integer ε such that for all perturbed motions $\Delta x(t)$, for which the inequality $|\Delta x(0)| \leqslant \delta$ is satisfied at initial moment $t = 0$, the following inequality will be satisfied:

$$|\Delta x(t)| < \varepsilon \tag{2.21}$$

for all $t > 0$.

If inequality (2.21) is satisfied for all initial perturbations which satisfy, in addition to the condition $|\Delta x(0) \leqslant \delta|$, additional conditions of the form

$$F(\Delta x(0)) = 0 \quad \text{or} \quad F(\Delta x(0)) \geqslant 0,$$

then the examined unperturbed motion is called *conditionally stable*. Motion stability in the sense of the basic definition is sometimes called *unconditional*.

Unperturbed motion $x_{out}^{(0)}(t)$ is called *asymptotically stable*, if, in addition to condition $|\Delta x(0)| \leqslant \delta$ and inequality (2.21), the equation

$$\lim_{t \to \infty} |\Delta x(t)| = 0. \tag{2.22}$$

is valid.

If the conditions of stability are satisfied for all $\Delta x(0)$, then we have *stability in the large*. If in order to satisfy them it is required that $\Delta x(0)$ be quite small, then we have *stability in the small*. /96

If for arbitrarily small $\delta(\varepsilon)$ there exists an ε such that it is possible to find a $\Delta x(0)$, satisfying condition $|\Delta x(0)| \leqslant \delta$ such that inequality (2.21) is not satisfied for some $t > 0$, then perturbed motion $x_{out}^{(0)}$ is called *unstable*.

In certain problems of aircraft dynamics it is more convenient from the practical point of view to state problems of stability in relation to the sum of the variables. Thus, for example, in control of angular motions of aircraft the change of velocity in perturbed motion is usually ignored and, in particular, the problem of the stability of unperturbed longitudinal motion is stated as the problem of stability in relation to only part of the variables, specifically in relation to α, ϑ and θ.

Sometimes it is necessary to choose the parameters such that the investigated unperturbed motion will be intentionally unstable. An example of such motion is the flight in spin.

The stated definition of stability shows that the solution of the problem of stability of unperturbed motion amounts to determining the behavior of the solution $\Delta x(t)$ of perturbed motion equations.

It is further assumed that perturbed motion is described by a system of linear differential equations with constant coefficients[4]. In this case solution of the problem of motion stability in terms of variable x_{out} amounts to determination of the sign of the roots of characteristic equation (see Section 2.1.4)

$$M(s)=0, \tag{2.23}$$

formed by equating to zero the denominator of the transfer function Y(s), written in terms of the corresponding variable.

In order that unperturbed motion $x_{out}^{(0)}(t)$ be asymptotically stable it is necessary and sufficient that all roots $s_1, \ldots, s_m$ of characteristic equation $M(s) = 0$ have negative real parts, i.e.,

$$\operatorname{Re} s_i<0 \ (i=1,\ldots, m), \tag{2.24}$$

where m is the degree of the characteristic equation.

If even one of roots $s_1, \ldots, s_m$ of equation M(s) = 0 has a positive real part, then unperturbed motion is *unstable*.

Cases when there is among the roots of equation M(s) = 0 at least one zero root or a pair of purely imaginary roots, are called *critical* (neutral). In critical cases of stability or instability perturbed motion cannot be judged on the basis of linear equations, and in order to solve the problem of stability it is necessary to analyze the equations of perturbed motion in their initial, nonlinear form (see Section 2.4.2.). /97

Condition (2.24) shows that to answer the question of stability it is not necessary to know the roots $s_1, \ldots, s_m$ of the characteristic equation, and it is important to know only the sign of their real parts. Methods of determining the sign of the real parts of the roots of the characteristic equation are called *stability criteria*. There are two groups of these criteria: algebraic stability criteria and frequency stability criteria.

---

[4]Methods of analysis of motion stability under more general assumptions are discussed in Section 2.4.

Algebraic stability criteria. The stability criteria which make it possible to determine the sign of the roots of characteristic equation (2.23), written in algebraic form

$$M(s)=a_m s^m+a_{m-1}s^{m-1}+\ldots+a_0=0, \tag{2.25}$$

are called algebraic stability criteria. This group of criteria includes those of A. Hurwitz, E. Routh and others. Hurwitz criterion is the most useful one.

Hurwitz Criterion. In order for all roots of the characteristic equation (2.25) to have negative real parts it is necessary and sufficient that the Hurwitz determinant Δ_m and all its diagonal minors Δ_1, Δ_2, ..., Δ_{m-1} have the same sign as coefficient a_0, where

$$\Delta_1=a_1;\quad \Delta_2=\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix};$$

$$\Delta_3=\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \tag{2.26}$$

$$\Delta_m=\begin{vmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & a_{m-2} & \ldots & a_m \end{vmatrix}$$

Corollary of the Hurwitz Criterion. In order that all roots of characteristic equation (2.25) have negative real parts it is necessary that all coefficients a_0, ..., a_m of equation (2.25) have the same sign.

For the characteristics of low-order equations ($m \leqslant 5$) the requirements of /98 the Hurwitz criterion are simplified and acquire the following form:

$$\left.\begin{array}{ll} \text{when} & m=1 \quad a_0>0;\ a_1>0; \\ \text{when} & m=2 \quad a_0>0;\ a_1>0;\ a_2>0; \\ \text{when} & m=3 \quad a_0>0;\ a_1>0;\ a_2>0;\ a_3>0; \\ & \qquad \Delta_2>0; \\ \text{when} & m=4 \quad a_0>0;\ a_1>0;\ \ldots,\ a_4>0; \\ & \qquad \Delta_3>0; \\ \text{when} & m=5 \quad a_0>0;\ a_1>0;\ \ldots,\ a_5>0; \\ & \qquad \Delta_2>0;\ \Delta_4>0. \end{array}\right\} \tag{2.27}$$

If it is difficult to use the algebraic criteria (for example, high-order characteristic equation), then the frequency criteria are used.

Frequency Stability Criteria. The stability criteria that make it possible to determine the sign of the roots of the characteristic equation according to the properties of the corresponding frequency characteristics, are called frequency stability criteria.

The expression for the frequency response $Y(j\omega)$ can be obtained from the expression for the transfer function $Y(s)$ by substituting $s = j\omega$ (see Section 2.1.5.). Here the characteristic polynomial $M(s)$ acquires the form

$$M(j\omega)=a_m(j\omega)^m+\ldots+a_0=U(\omega)+jV(\omega), \tag{2.28}$$

where the expression $M(j\omega) = U(\omega) + jV(\omega)$ is called the characteristic sector.

When frequency ω changes from $-\infty$ to $+\infty$ the end of characteristic vector $M(j\omega)$ describes on plane $U(\omega)$, $jV(\omega)$ a curve, called the *characteristic curve*, or *hodograph of vector* $M(j\omega)$, or *Mikhaylov's hodograph*. The characteristic curve is symmetric with respect to the real axis $U(\omega)$, and therefore the range of change of frequency ω can be bounded by the limits 0, $+\infty$.

Mikhaylov's frequency criterion is the most useful of the frequency group for analyzing motion stability.

Mikhaylov's Criterion. When the real parts of all roots of the characteristic equation are negative it is necessary and sufficient that the characteristic vector $M(j\omega)$ as frequency ω increases from 0 to $+\infty$, rotate by angle $m\pi/2$, or that the characteristic curve, as frequency ω changes from 0 to $+\infty$, starting with the positive real half-axis, pass consecutively, counterclockwise (i.e., in the opposite direction) m quadrants, where m is the order of the characteristic equation. /99

Examples of the behavior of the characteristic curves are illustrated in Figures 2.12 and 2.13.

Corollary of Mikhaylov's Criterion. When the real parts of all roots of the characteristic equation are negative it is necessary and sufficient that, simultaneously:

1) the roots of polynomials $U(\omega)$ and $V(\omega)$, i.e., the roots of the real and imaginary parts of the characteristic vector, be real and alternate in magnitude;

2) for $\omega = 0$ the conditions

$$U(0)>0;\quad V(0)=0;\quad \left.\frac{dV(\omega)}{d\omega}\right|_{\omega=0}>0. \tag{2.29}$$

are satisfied.

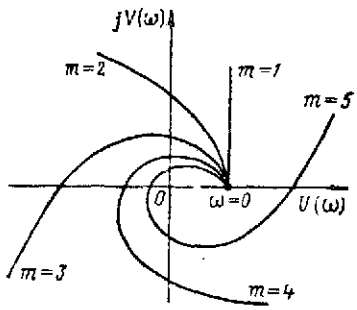


Figure 2.12. Behavior of Characteristic Curves of Aircraft Stability for Different Orders m of Characteristic Equation.

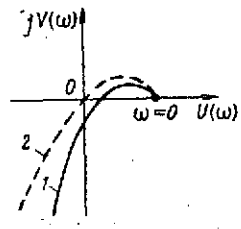


Figure 2.13. Behavior of the Characteristic Curve of Unstable Aircraft (Curve 1) and Aircraft on Boundary of Stability (Curve 2), for Third-Order Characteristic Equation (m = 3).

Graphic interpretation of the corollary of Mikhaylov's criterion is depicted in Figure 2.14.

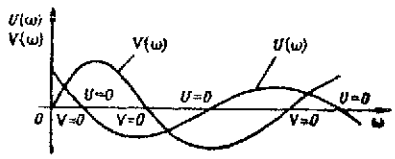


Figure 2.14. Example of Satisfying Requirements of Corollary of Mikhaylov's Criterion.

Construction of Regions of Stability

When analyzing motion stability the following question often arises: for what coefficients of characteristic equation M(s) = 0 is an aircraft stable and for what coefficients is it unstable? The answer to this question is found by plotting the boundaries of the region of stability for different coefficients $a_0, \ldots, a_m$ [see expression (2.25)]. From the total possible space of change of the stated coefficients it is necessary to distinguish the region of stability, i.e. the region in which the roots of the characteristic equation are negative or have negative real parts. The regions of stability can be constructed in the space of any number of coefficients. /100

Examined below is the case when the coefficients $a_0, \ldots, a_m$ depend linearly on any parameters A and B. The aerodynamic or design parameters of the aircraft can be regarded as such parameters.

Regions of stability in the plane of two real parameters A and B are constructed as follows. Characteristic equation (2.25) is written as

$$a_m(A, B)s^m + a_{m-1}(A, B)s^{m-1} + \ldots + a_0(A, B) = 0. \qquad (2.30)$$

Since coefficients $a_0, \ldots, a_m$ are linear functions of parameters A and B, equation (2.30) can be written as

$$AP(s)+BQ(s)+S(s)=0. \tag{2.31}$$

Substitution of $s = j\omega$, corresponding to the boundary of the region of stability, leads to the equation

$$[AP_1(\omega)+BQ_1(\omega)+S_1(\omega)]+j[AP_2(\omega)+BQ_2(\omega)+S_2(\omega)]=0. \tag{2.32}$$

where $P_1(\omega)$, $Q_1(\omega)$, $S_1(\omega)$ are the real parts of $P(j\omega)$, $Q(j\omega)$, $S(j\omega)$, respectively, and $P_2(\omega)$, $Q_2(\omega)$, $S_2(\omega)$ are the imaginary parts of $P(j\omega)$, $Q(j\omega)$, $S(j\omega)$, respectively.

Equation (2.32) reduces to two equations:

$$M_1(\omega)=AP_1(\omega)+BQ_1(\omega)+S_1(\omega)=0; \tag{2.33}$$
$$M_2(\omega)=AP_2(\omega)+BQ_2(\omega)+S_2(\omega)=0, \tag{2.34}$$

the solutions of which in terms of A and B are

$$A(\omega)=\Delta_A/\Delta; \quad B(\omega)=\Delta_B/\Delta, \tag{2.35}$$

where

$$\Delta=\begin{vmatrix} P_1(\omega) & Q_1(\omega) \\ P_2(\omega) & Q_2(\omega) \end{vmatrix}; \tag{2.36}$$

$$\Delta_A=\begin{vmatrix} -S_1(\omega) & Q_1(\omega) \\ -S_2(\omega) & Q_2(\omega) \end{vmatrix}; \tag{2.37}$$

$$\Delta_B=\begin{vmatrix} P_1(\omega) & -S_1(\omega) \\ P_2(\omega) & -S_2(\omega) \end{vmatrix}. \tag{2.38}$$

On the plane of A, B, each of equations $M_1(\omega) = 0$ and $M_2(\omega) = 0$, for some fixed $\omega = \omega^*$, describes a straight line. The coordinates of the point of intersection of these two lines represent the solution of equations $M_1(\omega) = 0$ and $M_2(\omega) = 0$ in terms of A and B for $\omega = \omega^*$. When ω changes from $-\infty$ to $+\infty$ a curve G is traced on plane A, B, which is the boundary of the region of stability (see Figures 2.15 and 2.16). /101

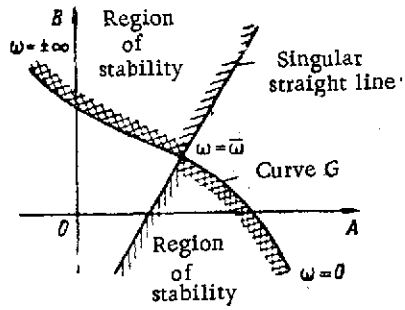


Figure 2.15. Example of Hatching of Boundary of Regions of Stability and Singular Straight Line. With passage through points $\omega = \bar{\omega}$ and $\omega = 0$ the determinant Δ changes sign.

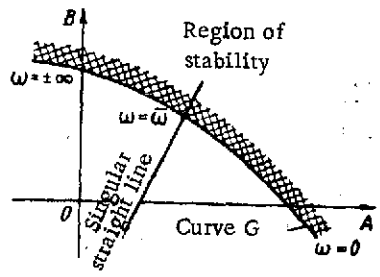


Figure 2.16. Example of Hatching of Boundary of Region of Stability. For $\omega < 0$, $\Delta < 0$; as point $\omega = \bar{\omega}$ is passed the sign of the determinant Δ is preserved; the singular straight line is not hatched.

Equations $M_1(\omega) = 0$ and $M_2(\omega) = 0$ determine one value of A and B for each ω only if they are compatible and independent, i.e., if the determinants Δ and Δ_A or Δ and Δ_B do not vanish simultaneously.

If for some $\omega = \bar{\omega}$, $\Delta = \Delta_A = 0$ or $\Delta = \Delta_B = 0$, the solutions of equations $M_1(\omega) = 0$ and $M_2(\omega) = 0$ are undefined, i.e., one equation is the consequence of the other. The above-mentioned lines then coincide, and consequently to all values of ω for which the determinants vanish correspond in plane A, B not a point, but a straight line, called the *singular line*.

To the points not lying on Curve G, given in parametric form $A = A(\omega)$, $B = B(\omega)$, correspond points of characteristic equation (2.32) with non-zero real parts. To denote the region in which the real parts of these roots are negative the G-boundary is cross-hatched.

Rule of Cross-Hatching. While moving on Curve G, as ω increases from $-\infty$ to $+\infty$, Curve G is cross-hatched on the left if $\Delta > 0$, and on the right if $\Delta < 0$.

The case when for some frequencies $\omega = \bar{\omega}$ Curve G intersects the line known as the *singular line*, is possible. The singular line is cross-hatched as shown in Figure 2.15 if as point $\omega = \bar{\omega}$ is passed the determinant Δ reverses signs, and is not cross-hatched if on the passage through point $\omega = \bar{\omega}$ the sign of Δ remains the same (see Figure 2.16). The region which becomes cross-hatched as the result of cross-hatching of Curve G may be the region of stability. This must be verified with the aid of some stability criterion, using any values $A = A^*$ and $B = B^*$ from this region.

When plotting the boundaries of stability regions parameter A from characteristic equation (2.31) is always located on the horizontal axis.

Stability Margins. A change of one or several coefficients a_i of characteristic equation (2.25) leads to a change of the magnitude and sign of the roots of the characteristic equation. The stated coefficients depend in general on the aerodynamic and design parameters of the aircraft, which are known or calculated with a certain accuracy. Therefore, theoretical satisfaction of the conditions of some stability criterion may not lead to actual motion stability. To avoid this situation it is necessary, in addition to the very fact of motion stability, to know how close the roots of the characteristic equation are on the complex plane to the imaginary axis or to the coordinate origin. The stated nearness is characterized by stability margins, which are used as a reserve of stability with respect to modulus and reserve of stability with respect to phase. /103

On the complex plane of distribution of the roots of the characteristic equation the *stability margin with respect to modulus h* is determined by the minimum value $|s_i|$ (i = 1, xxx, m) of the moduli of roots $|s_1|, \ldots, |s_m|$, where m is the order of the characteristic equation (Figure 2.17); *stability margin with respect to phase* γ is determined by the minimum difference

$$\min\left[|\arg s_i| - \frac{\pi}{2}\right] \quad (i = 1, \ldots, m),$$

where $|\arg s_i|$ is the absolute value of the argument of the i-th root of the m-order characteristic equation (see Figure 2.18).

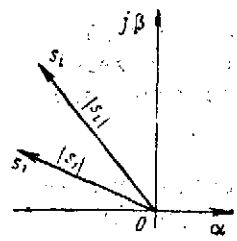


Figure 2.17. Determination of Stability Margin With Respect to Modulus h.

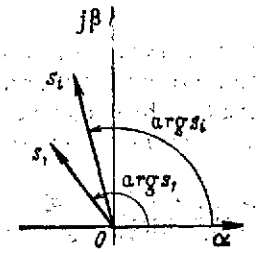


Figure 2.18. Determination of Stability Margin With Respect to Phase γ:

$$\gamma = \min\left\{|\arg s_1| - \frac{\pi}{2}; \ldots; |\arg s_m| - \frac{\pi}{2}\right\};$$

s_i is the i-th root of the characteristic equation: i = 1, ..., m.

The stability margins of a component (system) are meaningful only for a stable component (system). The stability margins of aircraft pertain to the most important characteristics of dynamic properties, since they influence speed, vibration, and damping properties.

2.2. Longitudinal Perturbed Motion of Aircraft

2.2.1. Equations of Motion

The linearized equations of longitudinal perturbed motion are written in the form

$$\begin{aligned}
&\frac{d\Delta V}{dt} - a_{11}\Delta V - a_{13}\Delta\theta - a_{14}\Delta\alpha = a_{16}\Delta\delta_в + a_{16}X_в;\\
&-a_{21}\Delta V + \frac{d^2\Delta\vartheta}{dt^2} - a_{22}\frac{d\Delta\vartheta}{dt} - a_{24}\Delta\alpha - a'_{24}\frac{d\Delta\alpha}{dt} = \\
&= a_{25}\Delta\delta_в + a'_{25}\frac{d\Delta\delta_в}{dt} + a_{26}M_{zв};\\
&-a_{31}\Delta V - a_{33}\Delta\theta + \frac{d\Delta\theta}{dt} - a_{34}\Delta\alpha = a_{35}\Delta\delta_в + a_{36}Y_в,
\end{aligned} \tag{2.39}$$

where

$$-\Delta\vartheta + \Delta\theta + \Delta\alpha = 0; \tag{2.40}$$

$$\frac{d\Delta\vartheta}{dt} = \Delta\omega_z. \tag{2.41}$$

In system (2.39) the independent, i.e., input variables are deviations ΔV, $\Delta\theta$, $\Delta\vartheta$ and $\Delta\alpha$; the output variables as known functions of time are control action $\Delta\delta_h$ and perturbations X_h, Y_h and M_{zh} (see Section 2.1.1.).

The zero solution of system (2.39) is: /104

$$\left.\begin{aligned}
&\Delta V = \Delta V^{(0)} = 0; \quad \Delta\theta = \Delta\theta^{(0)} = 0;\\
&\Delta\vartheta = \Delta\vartheta^{(0)} = 0; \quad \Delta\alpha = \Delta\alpha^{(0)} = 0,
\end{aligned}\right\} \tag{2.42}$$

and is valid for $\Delta\delta_h = 0$, $X_h = Y_h = 0$, $M_{zh} = 0$. It determines the longitudinal unperturbed motion of aircraft (see Table 2.1, Item 25).

The dynamic coefficients a_{ij} of system (2.39) correspond to unperturbed motion and are known functions of time.

Within the range of application of the "frozen coefficient" principle, the dynamic coefficients a_{ij} can be considered constant.

Aircraft motion, determined by the general solution of system (2.39), i.e., by the functions $\Delta V(t)$; $\Delta\vartheta(t)$, $\Delta\theta(t)$, $\Delta\alpha(t)$ is the sum of two motions: free and induced.

Free motion is determined by the general solution of system (2.39) for zero initial conditions, which can be regarded the result of pulsed actions at the initial moment of time $t = t_0$ (see Table 2.1, Item 18).

Induced motion is determined by the partial solution of system (2.39) for known functions $\Delta\delta_h(t)$, $X_h(t)$, $Y_h(t)$ or $M_{zh}(t)$. For known actions the character of induced motion is completely defined by the transfer functions of the aircraft.

2.2.2. Characteristic Polynomial and Characteristic Equation

Transformation of system (2.39) using relation (2.40), writing, for the transformed system the determinant

$$M(s)=\begin{vmatrix} s-a_{11} & 0 & -a_{13} & -a_{14} \\ -a_{21} & s(s-a_{22}) & 0 & -a_{24}-a'_{24}s \\ -a_{31} & 0 & s-a_{33} & -a_{34} \\ 0 & -1 & 1 & 1 \end{vmatrix} \tag{2.43}$$

and its solution lead to the characteristic polynomial

$$M(s)=s^4+A_1s^3+A_2s^2+A_3s+A_4, \tag{2.44}$$

where

$$\left.\begin{aligned} A_1&=-a_{33}+a_{34}-a_{22}-a'_{24}-a_{11};\\ A_2&=a_{21}a_{14}-a_{31}a_{13}+a_{22}a_{33}-a_{22}a_{34}-a_{24}+a_{33}a'_{24}+\\ &\quad+a_{33}a_{11}-a_{34}a_{11}+a_{22}a_{11}+a'_{24}a_{11};\\ A_3&=-a_{21}a_{14}-a_{31}a_{22}a_{14}+a_{22}a_{31}a_{13}+a'_{24}a_{31}a_{13}+\\ &\quad+a_{24}a_{33}-a_{22}a_{33}a_{11}+a_{22}a_{34}a_{11}+a_{24}a_{11}-a_{33}a_{11}a'_{24};\\ A_4&=a_{21}a_{33}a_{14}-a_{13}a_{21}a_{34}+a_{24}a_{31}a_{13}-a_{24}a_{33}a_{11}. \end{aligned}\right\} \tag{2.45}$$

The characteristic polynomial, equated to zero, forms the characteristic equation /105

$$M(s)=s^4+A_1s^3+A_2s^2+A_3s+A_4=0. \tag{2.46}$$

2.2.3. Free Motion

Analysis of free longitudinal motion amounts to analysis of the roots of characteristic equation (2.46) (see Table 2.5). The sign of the real parts of the roots determines the stability (or instability) of perturbed motion (2.42), and the form and magnitude of the roots determine the character of longitudinal free (perturbed) motion.

In the general case polynomial M(s) can be written as the product of two quadratic trinomials

$$M(s)=A_4(T^2s^2+2T\xi s+1)(\tau^2s^2+2\tau\eta s+1). \tag{2.47}$$

Typical for longitudinal perturbed motion is the presence among the roots of characteristic equation M(s) = 0, of two pairs of complex roots that differ by modulus: pairs of high modulus roots and pairs of low modulus roots. Trinomial $T^2s^2 + 2T\xi s + 1$, corresponding to the pair of large modulus roots, determines in summary free motion the fast component, or *short period motion*. Trinomial $\tau^2s^2 + 2\tau\eta s + 1$, corresponding to the pair of low modulus roots, determines in summary free motion the slow component, or *long period motion*.

Various methods of approximation can be used for determining the roots of the characteristic equation. The simplest of them, described in Table 2.5, reduces the computation of the two pairs of modulus-different roots to the solution, respectively, of the two quadratic equations:

$$s^2+(a_{34}-a_{22}-a'_{24})s+(-a_{24}-a_{22}a_{34})=0; \tag{2.48}$$

$$A_2s^2+A_3s+A_4=0. \tag{2.49}$$

Example 2. The example pertains to calculation of the "approximate" values of the roots of characteristic equation (2.46). Flight at an altitude of 12,000 m at 250 m/sec is examined; the dynamic coefficients a_{ij} of system (2.39) have the following values:

a_{11} = -0.0134 1/sec.; a_{14} = -9.35 m/sec.^{2};

a_{15} = 0; a_{13} = -9.81 m/sec.^{2};

a_{21} = 0; a_{22} = -0.49 1/sec.;

a_{24} = -2.98 1/sec.^{2}; a'_{24} = -0.22 1/sec.;

a_{25} = -2.32 1/sec.^{2}; a'_{25} = 0;

a_{31} = 0.0007 1/m; a_{34} = 0.577 1/sec.;

a_{35} = 0; a_{33} = 0;

consequently, the coefficients of the characteristic equation (2.46) are, according to (2.45)

$$A_1 = 1.299; \quad A_3 = 0.0455;$$
$$A_2 = 3.278; \quad A_4 = 0.0201.$$

Equation (2.48)

$$s^2 + 1.287s + 3.263 = 0$$

determines the following approximate values of the pair of high modulus roots:

$$s_{1,2} = -0.6435 \pm j\ 1.688.$$

Equation (2.49)

$$3.278\ s^2 + 0.0455s + 0.0201 = 0$$

determines the following approximate values of the pair of low modulus roots:

$$s_{3,4} = 0.0069 \pm 0.0780.$$

Here the exact roots are: $s_{1,2} = 0.644 \pm j\ 1.687;$

$$s_{3,4} = -0.0056 \pm j\ 0.0783.$$

The approximate roots, as is seen, are quite close to the exact roots.

Since short period motion is especially important for aircraft control it is helpful in many cases to distinguish this motion from the overall (summary) motion. For this purpose the change of velocity in equations (2.39) can be ignored (see Table 2.5), which reduces by one the order of the characteristic equation. If the effect of gravity on perturbed motion is also ignored (see Table 2.6), the order of the characteristic equation becomes two.

Analysis of the character of longitudinal free (perturbed) motion is presented in Tables 2.5 and 2.6. Also presented there are expressions for the damping coefficient, or logarithmic decrement ξ/T, for the frequency of free oscillations ω_{free} and for the frequency of free oscillations in the absence of damping (i.e., for $a_{34} - a_{22} - a_{24}' = 0$) of ω_a, called the natural frequency of oscillations of the aircraft.

2.2.4. Transfer Functions

Assuming the coefficients of the equations of system (2.39) to be constant, the Laplace transformation of the equation can be done for zero initial conditions:

$$\left.\begin{aligned}
&(s-a_{11})\Delta V(s)-a_{13}\Delta\theta(s)-a_{14}\Delta\alpha(s)=a_{15}\Delta\delta_{\text{h}}(s)+\\
&\qquad+a_{16}X_{\text{h}}(s);\\
&-a_{21}\Delta V(s)+s(s-a_{22})\Delta\vartheta(s)-a_{24}\Delta\alpha(s)-a'_{24}s\Delta\alpha(s)=\\
&\qquad=(a_{25}+a'_{25}s)\Delta\delta\text{h}(s)+a_{26}M_{z\,\text{h}}(s);\\
&-a_{31}\Delta V(s)-a_{33}\Delta\theta(s)+s\Delta\theta(s)-a_{34}\Delta\alpha(s)=\\
&\qquad=a_{35}\Delta\delta\text{h}(s)+a_{36}Y_{\text{h}}(s);\\
&-\Delta\vartheta(s)+\Delta\theta(s)+\Delta\alpha(s)=0.
\end{aligned}\right\} \tag{2.50}$$

Using equation system (2.50) it is possible to determine the transfer functions of the aircraft, determining its longitudinal perturbed motion (see Section 2.14). The transfer function can be found for each pair of input and output variables. The subscript in the symbols of the transfer functions corresponds to the input variable and the superscript to the output variable.

The solution, in particular, of algebraic equations (2.50) in terms of unknown $\Delta\varphi(s)$ leads to the expression

$$\Delta^{\vartheta}(s)=\frac{1}{M(s)}\times$$

$$\times\begin{vmatrix}
s-a_{11} & a_{15}\Delta\delta(s)+a_{16}X\text{h}(s) & -a_{13} & -a_{14}\\
-a_{21} & (a_{25}+a'_{25}s)\Delta\delta_{\text{h}}(s)+a_{26}M_{z\,\text{h}}(s) & 0 & -a_{24}-a'_{24}s\\
-a_{31} & a_{35}\Delta\delta\text{h}(s)+a_{36}Y\text{h}(s) & s-a_{33} & -a_{34}\\
0 & 0 & 1 & 1
\end{vmatrix} \tag{2.51}$$

where M(s) is the characteristic polynomial, determined by expression (2.46).

The solution of the determinant in (2.51) gives the following expression, which determines the total change of pitch in longitudinal perturbed motion:

$$\begin{aligned}
\Delta\vartheta(s)=&Y^{\vartheta}_{\delta}(s)\Delta\delta_{\text{h}}(s)+Y^{\vartheta}_{X}(s)\cdot X_{\text{h}}(s)+\\
&+Y^{\vartheta}_{Y}(s)Y_{\text{h}}(s)+Y^{\vartheta}_{M_z}(s)M_{z\,\text{h}}(s),
\end{aligned} \tag{2.52}$$

where the transfer functions $Y^{\vartheta}_{\delta}(s)$, $Y^{\vartheta}_{X}(s)$, $Y^{\vartheta}_{Y}(s)$ and $Y^{\vartheta}_{M_z}(s)$ characterize the reaction of pitch to the corresponding actions.

TABLE 2.5. ANALYSIS OF FREE LONGITUDINAL MOTION BASED ON CHARACTERISTIC POLYNOMIAL.

Form of Characteristic Polynomial	Composition of motion	Expansion of Characteristic Polynomial Into Factors	Expression for Roots of Characteristic Equation		Characteristics of Roots of Characteristic Equation
			Exact	Approximate	
$M(s)=s^4+A_1s^3+A_2s^2+$ $+A_3s+A_4$	The sum of long period (slow) motion and the short period (fast motion).	$M(s)=A_4(T^2s^2+2T\xi+1)\times$ $\times(\tau^2s^2+2\tau\eta s+1)$ $\tau=-4\left\{\frac{s_3}{(s_3+s_4)^3}\left(1-\right.\right.$ $\left.\left.-\frac{s_3}{s_3+s_4}\right)\right\}^{1/2}$ $\eta'=2\left[\frac{s_3}{s_3+s_4}\left(1-\frac{s_3}{s_3+s_4}\right)\right]^{1/2}$	$s_{1,2}=\frac{-\xi\pm\sqrt{\xi^2-1}}{T}$ $s_{3,4}=\frac{-\eta\pm\sqrt{\eta^2-1}}{\tau}$	$s_{1,2}=\frac{a_{22}+a'_{24}-a_{34}}{2}\pm$ $\pm\left[\frac{1}{4}(a_{34}-a_{22}-a'_{24})+\right.$ $\left.+a_{24}+a_{22}a_{34}\right]^{1/2}$ $s_{3,4}\approx$ $\approx\frac{-A_3\pm\sqrt{A_3^2-4A_2A}}{2A_2}$	Typical for aircraft is the case where the equation M(s) = 0 has two high-modulus roots (for example, s_1 and s_2), which determine short period motion, and two low-modulus roots (for example, s_3 and s_4), which determine long period motion in summary motion.
For expressions for A_1, A_2, A_3, A_4 see (2.45).		For T, ξ see Table 2.6.		For A_i see (2.45)	
$M_1(s)=s^3+A_5s^2+A_6s+A_7$ $A_5=-a_{22}-a'_{22}+a_{34}-a_{33}$ $A_6=-a_{24}-a_{22}a_{34}+$ $+(a_{22}+a'_{24})a_{33}$ $A_7=a_{24}a_{33}$ $M_1(s)\approx M(s)$ when $a_{11}\approx 0$ $a_{21}\approx 0;\ a_{31}\approx 0$ (for the case when $a_{33}\approx 0$, see Table 2.6)	Short period (fast) motion with reduced effect of long period (slow) motion due to neglect of the change in velocity.	$M_1(s)=(T^2s^2+2T\xi s+1)\times$ $\times(\tau s+1)$	$s_{1,2}=\frac{-\xi\pm\sqrt{\xi^2-1}}{T}$ $s_3=-\frac{1}{\tau}$	$s_{1,2}\approx-\frac{A_5}{2}\pm$ $\pm\sqrt{\frac{A_5^2}{4}-A_6}$ $s_3\approx\frac{A_7}{A_6}$ $\tau\approx-\frac{A_6}{A_7}$	The two high-modulus roots (s_1 and s_2) determine short period motion, but small root s_3 determines fraction of long period motion in summary motion. For certain types of aircraft (for example, for ballistic missiles) the typical case is $s_3>0$, i.e., $\tau<0$.

TABLE 2.6. ANALYSIS OF FREE LONGITUDINAL SHORT PERIOD MOTION BASED ON CHARACTERISTIC POLYNOMIAL.

Type of Motion and Form of Characteristic Polynomial	Nature of Unperturbed Motion for $(-a_{22} - a'_{24} + a_{34}) > 0$	Nature of the Transient Process	Factorization of Characteristic Polynomial			Expressions Needed for Calculating Roots According to Dynamic Coefficients of Aircraft
			Factorization Condition	Form of Factorization	Roots of Polynomial and their Characteristics	
Short period motion $M_2(s) = s^2 + A_8 s + A_9$ $A_8 = -a_{22} - a'_{24} + a_{34}$ $A_9 = -(a_{24} + a_{22}a_{34})$ $M_2(s) \approx M(s)$ when $a_{11} \approx 0$ $a_{21} \approx 0$ $a_{31} \approx 0$ $a_{33} \approx 0$	Stable $[(a_{24} + a_{22}a_{34}) < 0]$	Oscillatory	$\xi < 1$	$M_2(s) = T^2 s^2 + 2T\xi s + 1$	Two complex-conjugate roots with negative real parts $s_{1,2} = \frac{-\xi \pm j\sqrt{\xi^2 - 1}}{T}$	$T = \frac{1}{\sqrt{-a_{24} - a_{22}a_{34}}}$; $\xi = \frac{-a_{22} - a'_{24} + a_{34}}{2\sqrt{-a_{24} - a_{22}a_{34}}}$. $\xi \approx \frac{-a_{22} - a'_{24} + a_{34}}{2\sqrt{-a_{24}}}$ when $\lvert a_{24} \rvert \gg \lvert a_{22}a_{34} \rvert$
		Aperiodic	$\xi = 1$	$M_2(s) = (Ts + 1)^2$	Roots are multiple, real, negative $s_{1,2} = -\frac{1}{T}$	$T = (-a_{24} - a_{22}a_{34})^{-\frac{1}{2}}$
			$\xi > 1$	$M_2(s) = (T_1 s + 1) \times (T_2 s + 1)$ $T_1 \neq T_2$	Real roots, different in magnitude, negative $s_1 = -\frac{1}{T_1}$ $s_2 = -\frac{1}{T_2}$	$T_{1,2} = T(\xi \pm \sqrt{\xi^2 - 1})$ $T = \frac{1}{\sqrt{-a_{24} - a_{22}a_{34}}}$; $\xi = \frac{-a_{22} - a'_{24} + a_{34}}{2\sqrt{-a_{24} - a_{22}a_{34}}}$

TABLE 2.6. (CONTINUED)

Type of Motion and Form of Characteristic Polynomial	Nature of Unperturbed Motion for $(-a_{22} - a'_{24} + a_{34}) > 0$	Nature of Transient Process	Factorization of Characteristic Polynomial: Factorization Condition	Factorization of Characteristic Polynomial: Form of Factorization	Roots of Polynomial and their Characteristics	Expressions Needed for Calculating Roots According to Dynamic Coefficients of the Aircraft
(See Table 2.5)	Unstable $[(a_{21} + a_{22}a_{34}) > 0]$	Aperiodic	$\xi > 1$	$M_2(s) = (T_1 s + 1) \times (T_2 s + 1)$, $T_1 \neq T_2$	Real roots, one positive, the other negative: $s_1 = -\frac{1}{T_1}$, $s_2 = -\frac{1}{T_2}$	$T_{1,2} = \frac{1}{2}(\pm b + \sqrt{b^2 - 4a})$; $a = \frac{1}{-a_{24} - a_{22}a_{34}}$; $b = \frac{-a_{22} - a'_{24} + a_{34}}{-a_{24} - a_{22}a_{34}}$
	Neutral $[(a_{21} + a_{22}a_{34}) = 0]$			$M_2(s) = s(Ts + 1)$	One zero root, one negative: $s_1 = 0$; $s_2 = -\frac{1}{T}$	$T = (-a_{22} - a'_{24} + a_{34})^{-1}$

Note: For the oscillating transient process the following expressions are valid (see Table 2.2):

$$\omega_{free} = \sqrt{-a_{24} - a_{22}a_{34} - \frac{1}{4}(a_{34} - a_{22} - a'_{24})^2}$$

$\omega_a = \sqrt{-a_{24} - a_{22}a_{34}} \cong \sqrt{-a_{24}}$ when $|a_{24}| \gg |a_{22}a_{34}|$

$\frac{\xi}{T} = \frac{a_{34} - a_{22} - a'_{24}}{2}$ (see Table 2.2).

When constructing the structural diagrams of aircraft control systems the perturbing moment can be considered in accordance with the transfer function $Y^{\vartheta}_{M_z}(s)$ (see diagram "a" in Figure 2.19). But the perturbing moment can also be taken into account by reducing it to the angle of deflection of the control elements in accordance with the transfer function $Y^{\delta}_{M_z}(s)$ (see Table 2.7) (see diagram "b" in Figure 2.19). /112

If in an aircraft with the standard system deflection of the control elements has little effect on drag, lift, and slipstream lag moments, the coefficients a_{15}, a'_{25}, a_{35} can be ignored and the moment can be considered in accordance with the expression $Y^{\delta}_{M_z}(s) = a_{26}/a_{25}$, i.e., $\delta_{per} = a_{26}/(a_{25})M_{zh}$ (see diagram "c" in Figure 2.19).

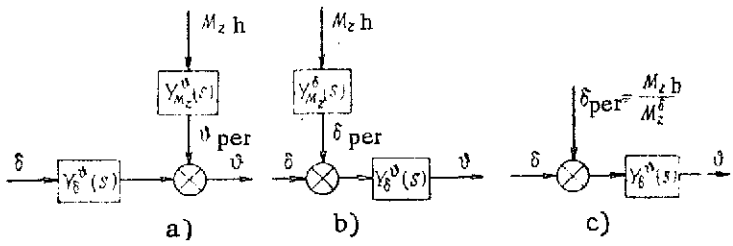


Figure 2.19. Ways of Accounting for Perturbing Moments When Drawing Structural Diagrams.

The transfer functions that determine the reaction of the various output variables to control and perturbing actions are presented in Tables 2.7, 2.7a and 2.7b. The transfer functions for short period motion (see Section 2.4.3.) of aircraft whose maximum lift, along with the normal component of thrust, significantly exceeds the weight of the aircraft (i.e., $a_{33} \approx 0$) are presented in Table 2.8.

The transfer functions of aircraft whose normal control force is small compared with wing lift are presented in Table 2.9.

The transfer coefficient, which represents the value of the transfer function for $s \to 0$ (see Table 2.1, item 20), is an important parameter of the transfer function. Different transfer coefficients of aircraft, determining the relation between the various parameters of the input and output variables in the steady state, are presented in Table 2.10.

TABLE 2.7: TRANSFER FUNCTIONS OF AIRCRAFT FOR TOTAL LONGITUDINAL MOTION

Δx_{in}, Δx_{out}	Symbol	General Form	Factored Form
$\Delta x_{in} \approx \Delta\delta_h$ $\Delta x_{out} = \Delta\vartheta$	$\frac{\Delta\vartheta(s)}{\Delta\delta_h(s)} = Y_\delta^\vartheta(s)$	$Y_\delta^\vartheta(s) = \frac{B_1 s^3 + B_2 s^2 + B_3 s + B_4}{s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4}$ For expressions for A_i see (2.45) $B_1 = a'_{25}$ $B_2 = a'_{25}(a_{34} - a_{11} - a_{33}) - a_{35}a'_{24} + a_{25}$ $B_3 = a'_{25}[a_{31}(a_{14} - a_{13}) - a_{11}(a_{34} - a_{33})] +$ $+ a_{15}(a_{21} - a_{31}a'_{24}) + a_{35}(a_{11}a'_{24} -$ $- a_{24}) + a_{25}(a_{34} - a_{11} - a_{33})$ $B_4 = a_{25}[a_{11}a_{24} - a_{21}(a_{14} - a_{13})] -$ $- a_{15}[a_{31}a_{24} - a_{21}(a_{34} - a_{33})] +$ $+ a_{25}[a_{31}(a_{14} - a_{13}) - a_{11}(a_{34} - a_{33})]$	$Y_\delta^\vartheta(s) = K_\delta^\vartheta \frac{(T_1 s + 1)(T_2 s + 1)(\tau_1 s + 1)}{(T^2 s^2 + 2T\xi s + 1)(\tau^2 s^2 + 2\tau\eta s + 1)}$ See Tables 2.5, 2.6 for definitions of T, τ, ξ and η. $K_\delta^\vartheta = \frac{B_4}{A_4}$; $\tau_1 \approx \frac{B_3}{B_4}$ $T_{1,2} = 2a'_{25}\{(a'_{25}a_{34} - a_{35}a'_{24} + a_{25}) \mp [(a'_{25}a_{34} -$ $- a_{35}a'_{24} + a_{25})^2 - 4a'_{25}(a_{25}a_{34} - a_{24}a_{35})]^{1/2}\}^{-1}$;
$\Delta x_{in} = M_{zh}$ $\Delta x_{out} = \Delta\vartheta$	$\frac{\Delta\vartheta(s)}{M_{zh}(s)} = Y_{M_z}^\vartheta(s)$	$Y_{M_z}^\vartheta(s) = \frac{E_1 s^2 + E_2 s + E_3}{s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4}$ For expressions for A_i see (2.45) $E_1 = a_{26}$ $E_2 = a_{26}(a_{34} - a_{11} - a_{33})$ $E_3 = a_{26}[a_{11}(a_{24} + a_{33}) +$ $+ a_{31}(a_{14} - a_{13})]$	$Y_{M_z}^\vartheta(s) = \frac{K_{M_z}^\vartheta(T_3 s + 1)(T_4 s + 1)}{(T^2 s^2 + 2T\xi s + 1)(\tau^2 s^2 + 2\tau\eta s + 1)}$ See Tables 2.5, 2.6 for definitions of T, τ, ξ and η. $K_{M_z}^\vartheta = \frac{E_3}{A_4}$; $T_{3,4} = 2\{a_{34} - a_{11} - a_{33} \mp [(a_{34} -$ $- a_{11} - a_{33})^2 - 4a_{11}(a_{24} + a_{33}) - 4a_{31}(a_{14} - a_{13})]^{1/2}\}^{-1}$;

TABLE 2.7. (CONTINUED)

Δx_{in}, Δx_{out}	Symbol	General Form	Factored Form
$\Delta x_{in} = M_{z_h}$ $\Delta x_{out} = \Delta\delta_h$	$\frac{\Delta\dot{\delta}_h(s)}{M_{zh}(s)} = Y^{\delta}_{M_z}(s)$	$Y^{\delta}_{M_z}(s) = \frac{E_1 s^2 + E_2 s + E_3}{B_1 s^3 + B_2 s^2 + B_3 s + B_4}$	$Y^{\delta}_{M_z}(s) = K^{\delta}_{M_z} \frac{(T_3 s + 1)(T_4 s + 1)}{(T_1 s + 1)(T_2 s + 1)(\tau_1 s + 1)}$; $K^{\delta}_{M_z} = \frac{E_3}{B_4}$

Note: The typical case for aircraft is when equation $B_1 s^3 + B_2 s^2 + B_3 s + B_4 = 0$ has two large-modulus roots (determined by parameters T_1 and T_2 and corresponding to short period motion) and one small-modulus real root (determined by parameter τ_1 and corresponding to long period motion).

TABLE 2.7a: AIRCRAFT TRANSFER FUNCTIONS FOR TOTAL LONGITUDINAL MOTION

Δx_{in}, Δx_{out}	Symbol	General Form
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\theta$	$\frac{\Delta\theta(s)}{\Delta\delta_h(s)} = Y_\delta^\theta(s)$	$Y_\delta^\theta(s) = \frac{C_1 s^3 + C_2 s^2 + C_3 s + C_4}{s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4}$ See (2.45) for expressions for A_i. $C_1 = a_{35}$ $C_2 = -a_{35}(a_{11} + a_{22} + a'_{24}) + a'_{25}a_{34} + a_{15}a_{34}$ $C_3 = a_{35}[a_{11}(a_{22} + a'_{24}) - a_{24}] + a_{23}a_{34} -$ $- a'_{25}(a_{11}a_{34} - a_{14}a_{31}) - a_{15}a_{31}(a_{22} + a'_{24})$ $C_4 = a_{35}(a_{11}a_{24} - a_{21}a_{14}) - a_{25}(a_{11}a_{34} -$ $- a_{14}a_{31}) - a_{15}(a_{31}a_{24} - a_{21}a_{34})$
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\dot{\theta}$	$\frac{\Delta\dot{\theta}(s)}{\Delta\delta_h(s)} = Y_\delta^{\dot\theta}(s)$	$Y_\delta^{\dot\theta}(s) = \frac{s(C_1 s^3 + C_2 s^2 + C_3 s + C_4)}{s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4}$ See (2.45) for expressions for A_i.
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\alpha$	$\frac{\Delta\alpha(s)}{\Delta\delta_h(s)} = Y_\delta^\alpha(s)$	$Y_\delta^\alpha(s) = \frac{D_1 s^3 + D_2 s^2 + D_3 s + D_4}{s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4}$ See (2.45) for expressions for A_i. $D_1 = a'_{25} - a_{35}$ $D_2 = -a_{35}(a_{22} - a_{11}) + a_{25} - a'_{25}(a_{11} +$ $+ a_{33}) - a_{15}a_{31}$ $D_3 = -a_{35}a_{11}a_{22} - a_{25}(a_{11} + a_{33}) -$ $- a'_{25}(a_{31}a_{13} - a_{11}a_{33}) + a_{15}(a_{31}a_{22} + a_{21})$ $D_4 = a_{35}a_{21}a_{13} - a_{25}(a_{13}a_{31} - a_{11}a_{33}) -$ $- a_{15}a_{21}a_{33}$

TABLE 2.7b: AIRCRAFT TRANSFER FUNCTIONS FOR SHORT PERIOD LONGITUDINAL MOTION

($a_{11} = a_{21} = a_{31} = 0$)

Δx_{in}, Δx_{out}	Symbol	General Form	Factored Form
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\vartheta$	$\frac{\Delta\vartheta(s)}{\Delta\delta_h(s)} = Y_\delta^\vartheta(s)$	$Y_\delta^\vartheta(s) = \frac{B_1 s^2 + B_5 s + B_6}{s^3 + A_5 s^2 + A_6 s + A_7}$ See Table 2.5 for expressions for A_i. $B_1 = a'_{25}$ $B_5 = a'_{25}(a_{34} - a_{33}) - a_{35}a'_{24} + a_{25}$ $B_6 = a_{25}(a_{34} - a_{33}) - a_{35}a_{21}$	$Y_\delta^\vartheta(s) = K\frac{(T_1 s + 1)(T_2 s + 1)}{(T^2 s^2 + 2T\xi s + 1)(\tau s + 1)}$ See Table 2.5, 2.6 for expressions for T, τ, ξ. $K = \frac{B_6}{A_7}$ $T_{1,2} = \frac{2B_1}{B_5 \mp \sqrt{B_5^2 - 4B_1B_6}}$
$\Delta x_{in} = M_{zh}$ $\Delta x_{out} = \Delta\vartheta$	$\frac{\Delta\vartheta(s)}{M_{z_h}(s)} = Y_{M_z}^\vartheta(s)$	$Y_{M_z}^\vartheta(s) = \frac{E_4 s + E_5}{s^3 + A_5 s^2 + A_6 s + A_7}$ See Table 2.5 for expressions for A_i. $E_4 = a_{26}$ $E_5 = a_{26}(a_{34} - a_{33})$	$Y_{M_z}^\vartheta(s) = K_{M_z}^\vartheta\frac{T_5 s + 1}{(T^2 s^2 + 2T\xi s + 1)(\tau s + 1)}$ See Table 2.5, 2.6 for expressions for T, τ, ξ. $K_{M_z}^\vartheta = \frac{E_5}{A_7}$; $T_5 = \frac{E_4}{E_5}$
$\Delta x_{in} = M_{zh}$ $\Delta x_{out} = \Delta\delta_h$	$\frac{\Delta\vartheta_\delta(s)}{M_{z_h}(s)} = Y_{M_z}^\delta(s)$	$Y_{M_z}^\delta(s) = \frac{E_4 s + E_5}{B_1 s^2 + B_5 s + B_6}$	$Y_{M_z}^\delta(s) = K_{M_z}^\delta\frac{T_5 s + 1}{(T_1 s + 1)(T_2 s + 1)}$; $K_{M_z}^\delta = \frac{E_5}{B_6}$ For the case $a_{15} = a'_{25} = a_{35} = 0$ $Y_{M_z}^\delta(s) = \frac{a_{26}}{a_{25}}$
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\theta$	$\frac{\Delta\vartheta(s)}{\Delta\delta_h(s)} = Y_\delta^\vartheta(s)$	$Y_\delta^\vartheta(s) = \frac{C_1 s^2 + C_5 s + C_6}{s^3 + A_5 s^2 + A_6 s + A_7}$ See Table 2.5 for expressions for A_i. $C_1 = a_{35}$ $C_5 = -a_{35}(a_{22} + a'_{24}) + a'_{25}a_{34}$ $C_6 = a_{34}a_{35} - a_{35}a_{24}$	$Y_\delta^\vartheta(s) = K_\delta^\vartheta\frac{T_\vartheta^2 s^2 + 2T_\vartheta\xi_\vartheta s + 1}{(T^2 s^2 + 2T\xi s + 1)(\tau s + 1)}$ See Table 2.5, 2.6 for expressions for T, τ, ξ. $K_\delta^\vartheta = \frac{C_6}{A_7}$ See Table 2.8 for expressions for T_ϑ, ξ_ϑ.

TABLE 2.7b: (CONTINUED)

Δx_{in}, Δx_{out}	Symbol	General Form	Factored Form
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\theta$	$\frac{\Delta\theta(s)}{\Delta\delta_h(s)} = Y_\delta^\theta(s)$	$Y_\delta^\theta(s) = \frac{s(C_1s^2 + C_5s + C_6)}{s^3 + A_5s^2 + A_6s + A_7}$ See Table 2.5 for expressions for A_i.	$Y_\delta^\theta(s) = K_\delta^\theta \frac{s(T_\theta^2 s^2 + 2T_\theta\xi_\theta s + 1)}{(T^2s^2 + 2T\xi s + 1)(\tau s + 1)}$ See Table 2.8 for expressions for T_θ, ξ_θ, and for T, τ, ξ see Table 2.5, 2.6.
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\alpha$	$\frac{\Delta\alpha(s)}{\Delta\delta_h(s)} = Y_\delta^\alpha(s)$	$Y_\delta^\alpha(s) = \frac{D_1s^2 + D_5s + D_6}{s^3 + A_5s^2 + A_6s + A_7}$ See Table 2.5 for expressions for A_i. $D_1 = a'_{25} - a_{35}$ $D_5 = a_{25} - a'_{25}a_{33} - a_{35}a_{22}$ $D_6 = -a_{25}a_{33}$	$Y_\delta^\alpha(s) = K_\delta^\alpha \frac{T_\alpha^2 s^2 + 2T_\alpha\xi_\alpha s + 1}{(T^2s^2 + 2T\xi s + 1)(\tau s + 1)}$ See Table 2.5, 2.6 for expressions for T, τ, ξ. $K_\delta^\alpha = \frac{D_6}{A_7}$; $T_\alpha^2 = \frac{D_1}{D_6}$ $\xi_\alpha = \frac{1}{2}\frac{D_5}{\sqrt{D_1D_6}}$

TABLE 2.8: AIRCRAFT TRANSFER FUNCTIONS FOR LONGITUDINAL SHORT PERIOD MOTION
($a_{11} = a_{21} = a_{31} = a_{33} = 0$)

Δx_{in}, Δx_{out}	General Form	Factored Form		Expressions for parameters of numerators of transfer functions through dynamic coefficients of aircraft (for parameters of denominators see Table 2.6).
		For aircraft of "duck" design with wings rotating around Oz_1 axis	For aircraft of standard design and "tailless" design	
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\vartheta$	$Y_\delta^\vartheta(s) = \dfrac{a'_{25}s + (a_{25} + a'_{25}a_{34} - a'_{24}a_{35})s - a_{24}a_{35} + a_{25}a_{34}}{s\left[s^2 + (a_{34} - a_{22} - a'_{24})s - a_{24} - a_{22}a_{34}\right]}$	$K\dfrac{(T_1s+1)(T_2s+1)}{s(T^2s^2+2T\xi s+1)}$	$K\dfrac{T_1s+1}{s(T^2s^2+2T\xi s+1)}$	$K = \dfrac{-a_{25}a_{31} + a_{24}a_{35}}{a_{21} + a_{22}a_{34}}$ $T_1 = B + \sqrt{A+B^2}$ $T_2 = B - \sqrt{A+B^2}$ $A = \dfrac{a'_{25}}{a_{24}a_{31} - a_{25}a_{34}}$ $B = \dfrac{a_{25} + a'_{25}a_{34} - a'_{24}a_{35}}{a_{25}a_{31} - a_{24}a_{35}}$ (see Table 2.10)
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\dot{\vartheta}$	$Y_\delta^{\dot\vartheta}(s) = \dfrac{a'_{25}s^2 + (a_{25} + a'_{25}a_{34} - a'_{24}a_{35})s - a_{24}a_{35} + a_{25}a_{34}}{s^2 + (a_{34} - a_{22} - a'_{24})s - a_{22}a_{34} - a_{24}}$	$K\dfrac{(T_1s+1)(T_2s+1)}{T^2s^2+2T\xi s+1}$	$K\dfrac{T_1s+1}{T^2s^2+2T\xi s+1}$	
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\alpha$	$Y_\delta^\alpha(s) = \dfrac{-(a_{34} - a'_{25})s + (a_{25} + a_{22}a_{35})}{s^2 + (a_{34} - a_{22} - a'_{24})s - a_{24} - a_{22}a_{34}}$	$K_\alpha\dfrac{T_6s+1}{T^2s^2+2T\xi s+1}$	$K_\alpha\dfrac{T_5s+1}{T_2s^2+2T\xi s+1}$	$K_\alpha = -\dfrac{a_{25} + a_{22}a_{35}}{a_{21} + a_{22}a_{34}}$ $T_5 = -\dfrac{a_{35}}{a_{25} + a_{22}a_{35}}$ $T_6 = \dfrac{a'_{25} - a_{35}}{a_{25} + a_{22}a_{35}}$ (see Table 2.10)
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\dot\theta$	$Y_\delta^{\dot\theta}(s) = \dfrac{a_{35}s^2 + (a'_{25}a_{34} - a_{22}a_{35} - a'_{24}a_{35})s - a_{24}a_{35} + a_{25}a_{34}}{s^2 + (a_{34} - a_{22} - a'_{24})s - a_{24} - a_{22}a_{34}}$	$K\dfrac{T_\theta^2s^2+2T_\theta\xi_\theta s+1}{T^2s^2+2T\xi s+1}$	$K\dfrac{(T_3s+1)(T_4s+1)}{T^2s^2+2T\xi s+1}$	$K = \dfrac{-a_{25}a_{34} + a_{24}a_{35}}{a_{24} + a_{22}a_{34}}$ $T_\theta = \sqrt{\dfrac{a_{35}}{a_{25}a_{34} - a_{24}a_{35}}}$ $\xi_\theta = \dfrac{a'_{25}a_{34} - a_{22}a_{35} - a'_{24}a_{35}}{2\sqrt{a_{35}(a_{25}a_{34} - a_{24}a_{35})}}$ $T_3 = D + \sqrt{C+D^2}$ $C = \dfrac{a_{35}}{a_{25}a_{34} - a_{25}a_{34}}$ $T_4 = D - \sqrt{C+D^2}$ $D = \dfrac{a_{35}(a_{22} + a'_{24})}{a_{24}a_{35} - a_{25}a_{34}}$ (see Table 2.10)
$\Delta x_{in} = \Delta\vartheta_h$ $\Delta x_{out} = \Delta\theta$	$Y_\delta^\theta(s) = \dfrac{a_{35}s^2 + (a'_{25}a_{34} - a_{22}a_{35} - a'_{24}a_{35})s - a_{24}a_{35} + a_{25}a_{34}}{s\left[s^2 + (a_{34} - a_{22} - a'_{24})s - a_{24} - a_{22}a_{34}\right]}$	$K\dfrac{T_\theta^2s^2+2T_\theta\xi_\theta s+1}{s(T^2s^2+2T\xi s+1)}$	$K\dfrac{(T_3s+1)(T_4s+1)}{s(T^2s^2+2T\xi s+1)}$	

TABLE 2.9: AIRCRAFT TRANSFER FUNCTIONS FOR LONGITUDINAL SHORT PERIOD MOTION

(when $a_{11} = a_{21} = a_{31} = a_{15} = a_{33} = a'_{25} = a_{35} = 0$)

Δx_{in}, Δx_{out}	Form of transfer function	Expression of parameters of transfer function through dynamic coefficients of the aircraft
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\vartheta$	$Y_\delta^\vartheta(s) = \dfrac{K(T_1 s + 1)}{s(T^2 s^2 + 2T\xi s + 1)}$	$K = -\dfrac{a_{25}a_{34}}{a_{24} + a_{22}a_{34}}$ $T_1 = \dfrac{1}{a_{34}}$ $T = \dfrac{1}{\sqrt{-(a_{24} + a_{22}a_{34})}}$ $\xi = \dfrac{a_{34} - a_{22} - a'_{24}}{2\sqrt{-(a_{24} + a_{22}a_{34})}}$ (see Table 2.6)
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\dot{\vartheta}$	$Y_\delta^{\dot\vartheta}(s) = \dfrac{K(T_1 s + 1)}{T^2 s^2 + 2T\xi s + 1}$	
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\vartheta$	$Y_\delta^\vartheta(s) = \dfrac{K}{s(T^2 s^2 + 2T\xi s + 1)}$	
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\dot{\vartheta}$	$Y_\delta^{\dot\vartheta}(s) = \dfrac{K}{T^2 s^2 + 2T\xi s + 1}$	
$\Delta x_{in} = \Delta\delta_h$ $\Delta x_{out} = \Delta\alpha$	$Y_\delta^\alpha(s) = \dfrac{KT_1}{T^2 s^2 + 2T\xi s + 1}$	
$\Delta x_{in} = \Delta v_B$ $\Delta x_{out} = \Delta n_y$	$Y_\delta^{n_y}(s) = \dfrac{K\frac{V}{g}}{T^2 s^2 + 2T\xi s + 1}$	

TABLE 2.10: AIRCRAFT TRANSFER FUNCTIONS FOR LONGITUDINAL SHORT PERIOD MOTION

Name	Symbol	Definition	Relation of transfer coefficient to aircraft dynamic coefficients	
			For Stable Aircraft ($a_{24} + a_{22}a_{34} < 0$)	For Neutral Aircraft ($a_{24} + a_{22}a_{34} = 0$)
Aircraft transfer coefficient for angular velocity	K	$K = \left(\frac{\Delta\dot{\theta}}{\Delta\delta_h}\right)_{st}$ $K = \left(\frac{\Delta\dot{\theta}}{\Delta\delta_h}\right)_{st}$	$K = \frac{-a_{25}a_{34} + a_{24}a_{35}}{a_{24} + a_{22}a_{34}}$ $K \approx a_{35} - \frac{a_{25}}{a_{24}}a_{34}$ when $\lvert a_{22}a_{34}\rvert \ll \lvert a_{24}\rvert$	$K = \frac{a_{25}a_{31} - a_{24}a_{35}}{a_{31} - a_{22} - a_{24}}$
Aircraft transfer coefficient for normal acceleration	KV	$KV = \left(\frac{f_y}{\Delta\delta_h}\right)_{st}$		
Aircraft transfer coefficient for normal overload	$K\frac{V}{g}$	$K\frac{V}{g} = \left(\frac{\Delta n_y}{\Delta\delta_h}\right)_{st}$		
Aircraft transfer coefficient for attack angle	K_α	$K_\alpha = \left(\frac{\Delta\alpha}{\Delta\delta_h}\right)_{st}$	$K_\alpha = -\frac{a_{25} + a_{22}a_{35}}{a_{24} + a_{22}a_{34}}$ $K_\alpha \approx -\frac{a_{25}}{a_{24}}$ when $\lvert a_{22}a_{35}\rvert \ll \lvert a_{25}\rvert$ $\lvert a_{22}a_{34}\rvert \ll \lvert a_{24}\rvert$	$K_\alpha = \frac{a_{25} + a_{22}a_{35}}{a_{31} - a_{22} - a_{24}}$

The sign of the transfer functions is determined by the aircraft design and by aerodynamic and design characteristics. Thus, for example, for $m_z^\alpha < 0$ aircraft of the "duck" design have $K_\alpha > 0$, and aircraft of standard design and of the "tailless" design have $K_\alpha < 0$. When formulating the frequency characteristics of an aircraft (see Section 2.2.5), its transfer coefficient is considered positive, and the minus sign, if the coefficient is negative, is taken into account when the structural diagrams of aircraft control systems are drafted.

The transfer functions of aircraft characterize its maneuverability. Thus, /122
in particular, they show that as a result of a step deflection of the control elements by angle $\Delta\delta_h$, on completion of the transient process (see Table 2.1, item 22), constant angular velocities $\Delta\dot{\vartheta}$, $\Delta\dot{\theta}$ and normal g-forces Δn_y are established. The following relations are valid:

$$(\Delta\dot{\vartheta})_{st} = (\Delta\dot{\theta})_{st} = K\Delta\delta_h; \tag{2.53}$$

$$(\Delta n_y)_{st} = K\frac{V}{g}\Delta\delta_h, \tag{2.54}$$

and the maximum possible steady state values of angular velocity and normal g-force are:

$$(\Delta\dot{\theta})_{max} = K(\Delta\delta_h)_{max}; \tag{2.55}$$

$$(\Delta n_y)_{max} = K\frac{V}{g}(\Delta\delta_h)_{max}, \tag{2.56}$$

where $(\Delta\delta_h)_{max}$ is the maximum possible angle of deflection of the control elements which does not violate the linearity of the initial equations of perturbed motion (2.39).

2.2.5. Frequency Responses

The transfer functions of aircraft are such that the cofactor corresponding to the transfer function of an oscillatory component enters in all other transfer functions of the aircraft. The frequency responses of an oscillatory component (Figure 2.20 and 2.21) show that in the case of weak damping ($\xi \ll 1$) and frequencies of control element deflection ω, close to the natural frequency of oscillations ω_a (see Table 2.6), resonance occurs, where the amplitude of oscillation becomes maximum when

$$\omega/\omega_a = \sqrt{1-2\xi^2}. \tag{2.57}$$

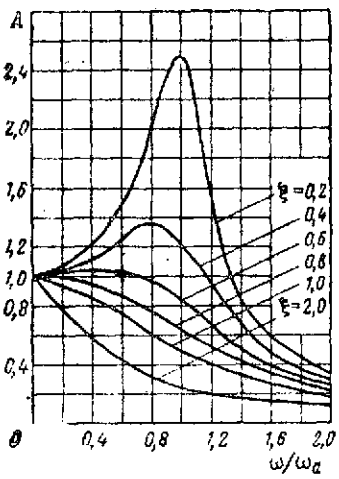


Figure 2.20. Amplitude--Frequency Responses of Oscillatory Component.

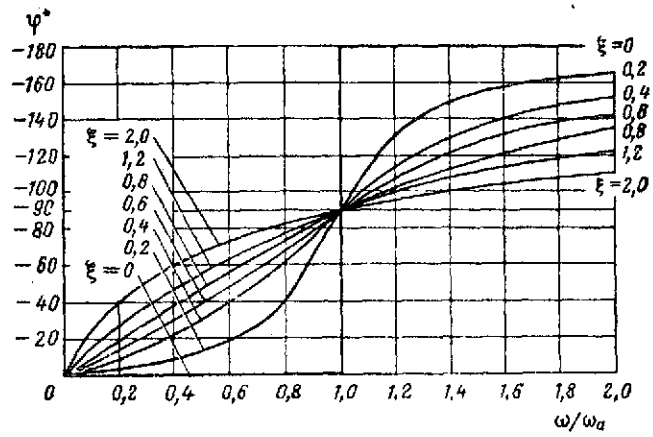


Figure 2.21. Phase-Frequency Responses of Oscillatory Component.

Since resonance is strongest for small ξ, the natural frequency ω_a, equal to

$$\omega_a = \sqrt{-(a_{23} + a_{22}a_{34})}. \qquad (2.58)$$

is usually assumed as the resonance frequency.

The transfer functions of aircraft of the type $Y_\delta^\alpha(s)$, $Y_\delta^{\dot\theta}(s)$ and $Y_\delta^{n_y}(s)$ are themselves the transfer functions of an oscillatory component. Examples of logarithmic frequency responses corresponding to these transfer functions are illustrated in Figure 2.22. Examples of frequency responses corresponding to transfer functions presented in Table 2.7, 2.7a, 2.8 and 2.9, are depicted in Figures 2.23-2.30.

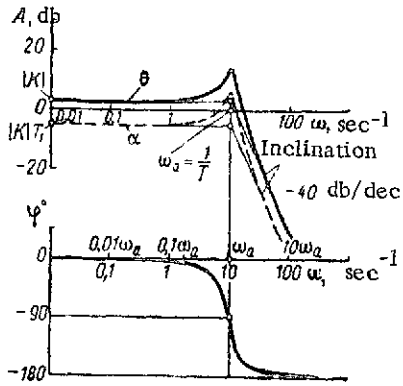


Figure 2.22. Logarithmic Frequency Responses of Transfer Functions $\theta(s)/\delta(s)$ and $\alpha(s)/\delta(s)$:

$|K| = 1.4$; $T = 0.1$ sec.; $\xi = 0.15$; $T_1 = 0.42$ sec.

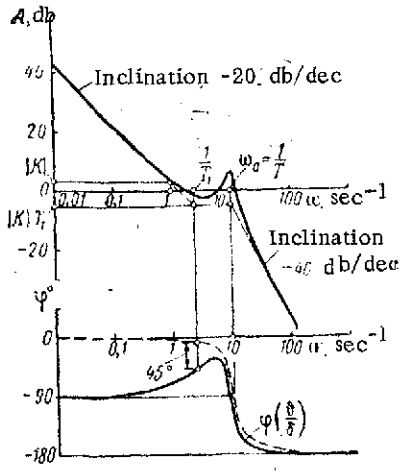


Figure 2.23. Logarithmic Frequency Responses of Transfer Function $\vartheta(s)/\delta(s)$:
$|K| = 1.4$; $T = 0.1$ sec;
$\xi = 0.15$; $T_1 = 0.42$ sec.

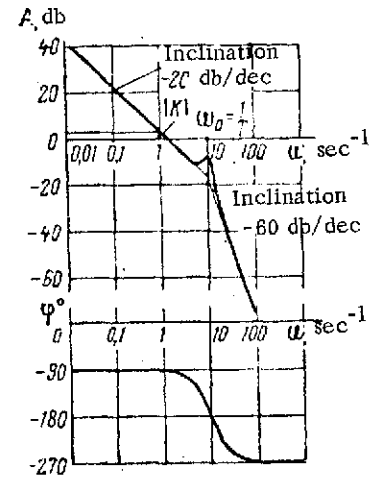


Figure 2.24. Logarithmic Frequency Responses of Transfer Function $\theta(s)/\delta(s)$:
$|K| = 1.4$; $T = 0.1$ sec;
$\xi = 0.15$.

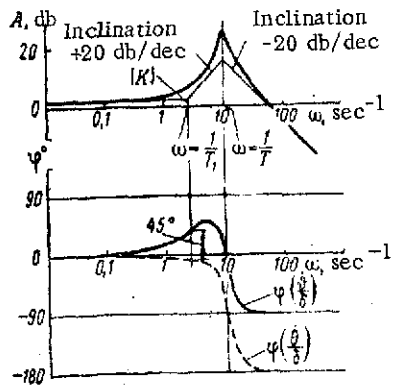


Figure 2.25. Logarithmic Frequency Responses of Transfer Function $\vartheta(s)/\delta(s)$:
$|K| = 1.4$; $T = 0.1$ sec;
$\xi = 0.15$; $T_1 = 0.42$ sec.

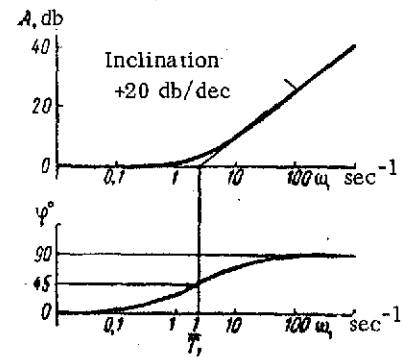


Figure 2.26. Logarithmic Frequency Responses of the Factor $T_1 s + 1$:
$T_1 = 0.42$ sec.

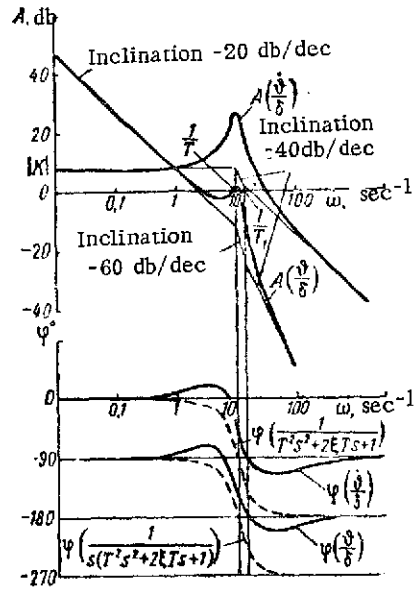


Figure 2.27. Logarithmic Frequency Responses of Transfer Functions $\vartheta(s)/\delta(s)$ and $\vartheta(s)/\delta(s)$ for Pivot-Wing Aircraft:
$|K| = 2.38$; $T = 0.1$ sec.;
$\xi = 0.15$; $T_1 = 0.084$ sec.

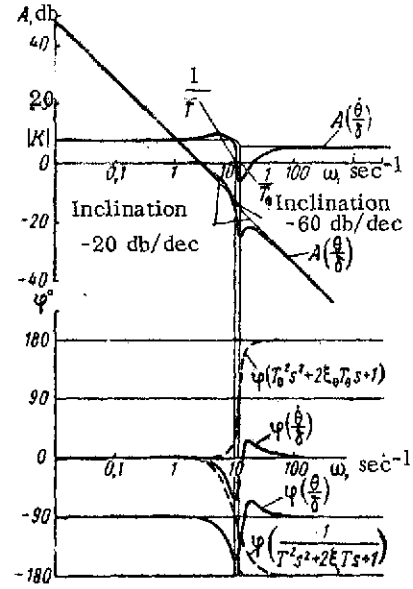


Figure 2.28. Logarithmic Frequency Responses of Transfer Functions $\vartheta(s)/\delta(s)$ and $\theta(s)/\delta(s)$ for Pivot-Wing Aircraft:
$|K| = 2.38$; $T = 0.1$ sec.;
$\xi = 0.15$; $T_\theta = 0.09$ sec.;
$\xi_\theta = 0.05$.

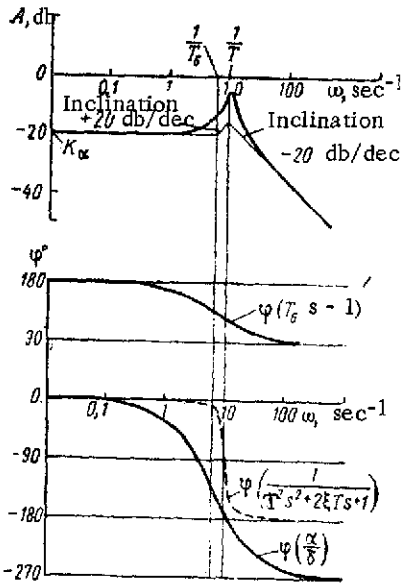


Figure 2.29. Logarithmic Frequency Responses for Transfer Function $\alpha(s)/\delta(s)$ for Pivot-Wing Aircraft:
$K_\alpha = 0.1$; $T = 0.1$ sec.; $\xi = 0.15$; $T_\theta = 0.125$.

2.2.6. Transient Functions

The transient functions of an aircraft (see Table 2.1, item 23), which determine the behavior of the various output variables in short period motion, caused by step-by-step deflection of the control elements, are given in Tables 2.11 and 2.12. /127

TABLE 2.11: AIRCRAFT TRANSFER FUNCTIONS FOR ϑ, $\dot{\vartheta}$ and θ when $\Delta\delta_h = [1]$, $\xi < 1$

Name	Symbol	Expressions for transfer function h(t)
Transfer function for angle of pitch	$h(t)=\frac{\Delta\vartheta}{K\Delta\delta}$	$\frac{\Delta\vartheta}{K\Delta\delta}=T\left[\frac{t}{T}-2\xi+\frac{T_1}{T}-e^{-\frac{\xi}{T}t}\sqrt{\frac{1-2\xi\frac{T_1}{T}+\left(\frac{T_1}{T}\right)^2}{1-\xi^2}}\times\sin\left(\frac{\sqrt{1-\xi^2}}{T}t+\varphi_2\right)\right]$ $\tan\varphi_2=\frac{\sqrt{1-\xi^2}\left(\frac{T_1}{T}-2\xi\right)}{1-2\xi^2+\xi\frac{T_1}{T}}$
Transfer function for angular velocity of pitch	$h(t)=\frac{\Delta\dot{\vartheta}}{K\Delta\delta}$	$\frac{\Delta\dot{\vartheta}}{K\Delta\delta}=1-e^{-\frac{\xi}{T}t}\sqrt{\frac{1-2\xi\frac{T_1}{T}+\left(\frac{T_1}{T}\right)^2}{1-\xi^2}}\times\cos\left(\frac{\sqrt{1-\xi^2}}{T}t+\varphi_1+\varphi_2\right)$ $\tan(\varphi_1+\varphi_2)=\frac{\frac{T_1}{T}-\xi}{\sqrt{1-\xi^2}}$
Transfer function for angle of inclination of trajectory	$h(t)=\frac{\Delta\theta}{K\Delta\delta}$	$\frac{\Delta\theta}{K\Delta\delta}=T\left[\frac{t}{T}-2\xi-\frac{e^{-\frac{\xi}{T}t}}{\sqrt{1-\xi^2}}\times\sin\left(\frac{\sqrt{1-\xi^2}}{T}t-2\varphi_1\right)\right]$ $\tan 2\varphi_1=\frac{2\xi\sqrt{1-\xi^2}}{1-2\xi^2}$

TABLE 2.12: AIRCRAFT TRANSFER FUNCTIONS FOR α, $\dot{\theta}$ and n_y WHEN $\Delta\delta_h$ = [1]

Name	Symbol	Oscillatory transient process ($\xi < 1$)		Aperiodic transient process ($\xi \geqslant 1$)	
		Expression for h(t)	Certain quality Criteria	Expression for h(t)	Certain quality Criteria
Transfer function for attack angle	$h(t)=\frac{\Delta\alpha}{KT_1\Delta\delta_h}$	$\frac{\Delta\alpha}{KT_1\Delta\delta_h}=1-\frac{e^{-\frac{\xi}{T}t}}{\sqrt{1-\xi^2}}\times$ $\times\cos\left(\frac{\sqrt{1-\xi^2}}{T}t-\varphi_1\right)$ $\tan\varphi_1=\frac{\xi}{\sqrt{1-\xi^2}}$	$\Delta\alpha_{max}=\Delta\alpha_{st}\ (1+\sigma)$; $\sigma=e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$ $\Delta\alpha_{st}=KT_1\Delta\delta$	$\frac{\Delta\alpha}{KT_1\Delta\delta}=1-$ $-\left(1+\frac{\xi}{T}t\right)e^{-\frac{\xi}{T}t}$	$\Delta\alpha_{max}=\Delta\alpha_{st}=$ $=KT_1\Delta\delta$
Transfer function for angular velocity of tangent to trajectory	$h(t)=\frac{\Delta\dot{\theta}}{K\Delta\delta_h}$	$\frac{\Delta\dot{\theta}}{K\Delta\delta_h}=1-\frac{e^{-\frac{\xi}{T}t}}{\sqrt{1-\xi^2}}\times$ $\times\cos\left(\frac{\sqrt{1-\xi^2}}{T}t-\varphi_1\right)$ $\tan\varphi_1=\frac{\xi}{\sqrt{1-\xi^2}}$	$\Delta\dot{\theta}_{max}=\Delta\dot{\theta}_{st}\ (1+\sigma)$; $\sigma=e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$ $\Delta\dot{\theta}_{st}=K\Delta\delta$	$\frac{\Delta\dot{\theta}}{K\Delta\delta}=1-$ $-\left(1+\frac{\xi}{T}t\right)e^{-\frac{\xi}{T}t}$	$\Delta\dot{\theta}_{max}=\Delta\dot{\theta}_{st}=$ $=K\Delta\delta$
Transfer function for normal g-force	$h(t)=\frac{\Delta n_y}{K\frac{V}{g}\Delta\delta_h}$	$\frac{\Delta n_y}{K\frac{V}{g}\Delta\delta_h}=1-\frac{e^{-\frac{\xi}{T}t}}{\sqrt{1-\xi^2}}\times$ $\times\cos\left(\frac{\sqrt{1-\xi^2}}{T}t-\varphi_1\right)$ $\tan\varphi_1=\frac{\xi}{\sqrt{1-\xi^2}}$	$\Delta n_{y\,max}=\Delta n_{y\,st}\ (1+\sigma)$; $\sigma=e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$ $\Delta n_{y\,st}=K\frac{V}{g}\Delta\delta$	$\frac{\Delta n_y}{K\frac{V}{g}\Delta\delta}=1-$ $-\left(1+\frac{\xi}{T}t\right)e^{-\frac{\xi}{T}t}$	$\Delta n_{y\,max}=\Delta n_{y\,st}=$ $=K\frac{V}{g}\Delta\delta$

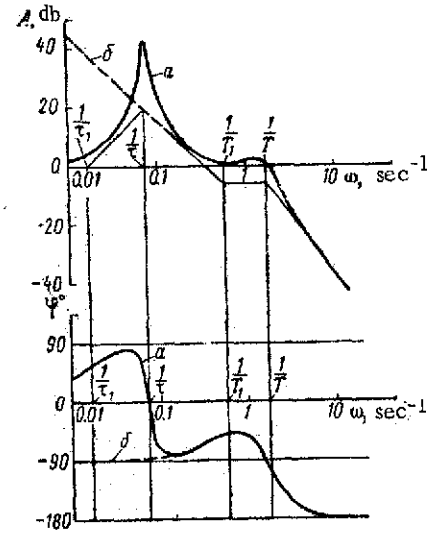


Figure 2.30. Logarithmic Frequency Responses of Transfer Function $\vartheta(s)/\delta(s)$: $T = 0.55$ sec; $\xi = 0.35$; $T_1 = 1.76$ sec; $T_2 = 0$; $\tau = 12.7$ sec; $\xi_\tau = 0.07$; $\tau = 71.4$ sec. a, Complete frequency responses; b, For assumptions $\Delta V = 0$ and $a_{44}\Delta\theta = 0$.

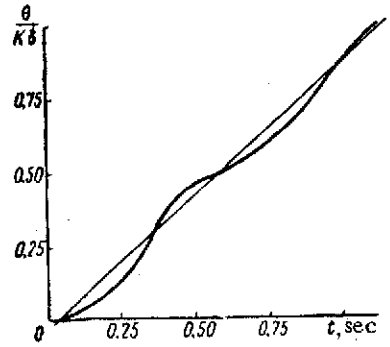


Figure 2.32. Change of Angle θ in Transient Process.

Examples of transient processes which correspond to certain transfer functions in Tables 2.11 and 2.12 are given in Figures 2.31-2.33. /131

As these relations show, transient processes in terms of $\Delta\alpha$, $\Delta\dot{\theta}$ and Δn_y differ for different relative damping coefficient values ξ. For a given ξ the time of transient process T_{con} (see Table 2.2) is inversely proportional to the natural frequency ω_a (see Table 2.6). The shortest transient process is maintained when $\xi = 0.75$. In this case the duration of the process is equal to

$$T_{con} \approx 3T \approx 3/\omega_a. \qquad (2.59)$$

Expressions for T, T_1, ξ and K, which make up the values of the transfer functions in Table 2.11, are given in Tables 2.6-2.10.

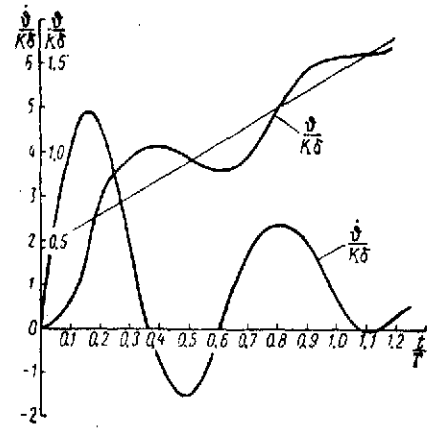


Figure 2.31. Change of Angle ϑ and Angular Velocity $\dot{\vartheta}$ in the Transient Process.

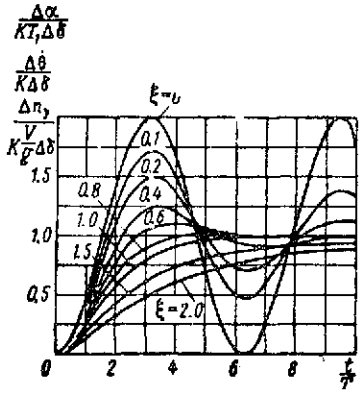


Figure 2.33. Nature of Transient Process $\dot{\vartheta}(t)$, $n_y(t)$ and $\alpha(t)$ as Function of Relative Coefficient of Damping ξ.

2.3. Lateral Perturbed Motion

2.3.1. Equations of Motion

Linearized equations of lateral perturbed motion of aircraft are represented in the form

$$\left.\begin{aligned}
&\frac{d\Delta\omega_x}{dt} - b_{11}\Delta\omega_x - b_{12}\Delta\omega_y - b_{16}\Delta\beta = \\
&\quad = b_{17}\Delta\delta_d + b_{18}\Delta\delta_e + b_{19}M_{xh}; \\
&-b_{21}\Delta\omega_x + \frac{d\Delta\omega_y}{dt} - b_{22}\Delta\omega_y - b_{26}\Delta\beta - b'_{24}\frac{d\Delta\beta}{dt} = \\
&\quad = b_{27}\Delta\delta_d + b'_{27}\frac{d\Delta\delta_d}{dt} + b_{29}M_{yh}; \\
&\frac{d\Delta\psi_V}{dt} - (b_{34} + b'_{34})\Delta\beta - b_{35}\Delta\gamma = b_{37}\Delta\delta_d + b_{39}Z_h; \\
&\left(\frac{1}{\cos\vartheta}\right)^{(0)}\Delta\omega_y - \frac{d\Delta\psi}{dt} = 0; \\
&\Delta\omega_x - (\tan\vartheta)^{(0)}\Delta\omega_y - \frac{d\Delta\gamma}{dt} = 0,
\end{aligned}\right\} \qquad (2.60)$$

where

$$(\cos\theta)^{(0)}\Delta\psi + \Delta\beta - (\sin\alpha)^{(0)}\Delta\gamma - (\cos\theta)^{(0)}\Delta\psi = 0. \qquad (2.61)$$

In system (2.60) the unknowns, i.e., the output variables, are deviations $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\psi_V$, $\Delta\beta$, $\Delta\gamma$, $\Delta\psi$; the input variables as known functions for time are the control actions $\Delta\delta_d$, $\Delta\delta_a$ and the perturbations M_{xh}, M_{yh}, Z_h (see Section 2.1.1.).

The zero solution of system (2.60)-(2.61) is /132

$$\left.\begin{aligned}
&\Delta\omega_x = \Delta\omega_x^{(0)} = 0; \quad \Delta\beta = \Delta\beta^{(0)} = 0; \\
&\Delta\omega_y = \Delta\omega_y^{(0)} = 0; \quad \Delta\gamma = \Delta\gamma^{(0)} = 0; \\
&\Delta\psi_V = \Delta\psi_V^{(0)} = 0; \quad \Delta\psi = \Delta\psi^{(0)} = 0,
\end{aligned}\right\} \qquad (2.62)$$

which is valid for $\Delta\delta_d = \Delta\delta_a = 0$; $Z_h = 0$; $M_{xh} = M_{yh} = 0$, determines lateral unperturbed motion (see Table 2.1, item 25).

All coefficients of equations (2.60)-(2.61), including the *dynamic coefficients* b_{ij}, correspond to unperturbed motion and are known time functions. Within limits of the applicability of the "frozen coefficients" principle they can be considered constant.

Aircraft motion which is determined by the general solution of system (2.60), i.e., by the functions $\Delta\omega_x(t)$, $\Delta\omega_y(t)$, $\Delta\psi_v(t)$, $\Delta\beta(t)$, $\Delta\gamma(t)$, $\Delta\psi(t)$, is the sum of two motions — free and induced.

Free motion is determined by the general solution of system (2.60) with non-zero initial conditions, which can be viewed as the result of pulse actions at the initial moment of time $t = t_0$ (see Table 2.1, item 18).

Induced motion is determined by the partial solution of system (2.60) for given functions $\Delta\delta_d(t)$, $\Delta\delta_a(t)$, $Z_h(t)$, $M_{xh}(t)$, $M_{yh}(t)$. With the given actions the nature of induced motion is completely determined by the aircraft transfer functions.

2.3.2. The Characteristic Polynomial and the Characteristic Equation

Transformation of system (2.60) using relationship (2.61) reduces the system to the form

$$\left.\begin{aligned}
&\frac{d\Delta\omega_x}{dt} - b_{11}\Delta\omega_x - b_{12}\Delta\omega_y - b_{14}\Delta\beta = \\
&\quad = b_{17}\Delta\delta_d + b_{18}\Delta\delta_e + b_{19}M_{xh};\\
&-b_{21}\Delta\omega_x + \frac{d\Delta\omega_y}{dt} - b_{22}\Delta\omega_y - b_{24}\Delta\beta - b'_{24}\frac{d\Delta\beta}{dt} = \\
&\quad = b_{27}\Delta\delta_d + b'_{27}\frac{d\Delta\delta_d}{dt} + b_{29}M_{yh};\\
&\left(\frac{1}{\cos\vartheta}\right)^{(0)}\Delta\omega_y - (b_{34}+b'_{34})\Delta\beta - \left(\frac{1}{\cos\theta}\right)^{(0)}\frac{d\Delta\beta}{dt} + \\
&\quad + \left(\frac{\sin\alpha}{\cos\vartheta}\right)^{(0)}\frac{d\Delta\gamma}{dt} - b_{35}\Delta\gamma = b_{37}\Delta\delta_d + b_{39}Z_h;\\
&\Delta\omega_x - (\tan\vartheta)^{(0)}\Delta\omega_y - \frac{d\Delta\gamma}{dt} = 0;\\
&\left(\frac{1}{\cos\vartheta}\right)^{(0)}\Delta\omega_y - \frac{d\Delta\psi}{dt} = 0.
\end{aligned}\right\} \qquad (2.63)$$

The latter equation does not have to be considered because the increase of $\Delta\psi$ does not enter into the remaining equations and the equation for $d\Delta\psi/dt$ can be integrated after solution of the remaining equations. /133

Compilation for the first four equations of system (2.63) of the determinant

$$M(s) = \begin{vmatrix} s-b_{11} & -b_{12} & -b_{14} & 0 \\ -b_{21} & s-b_{22} & -b_{24}-b'_{24}s & 0 \\ 0 & \left(\frac{1}{\cos\vartheta}\right)^{(0)} & -(b_{34}+b'_{34}) - \left(\frac{1}{\cos\theta}\right)^{(0)}s & \left(\frac{\sin\alpha}{\cos\vartheta}\right)^{(0)}s - b_{35} \\ 1 & -(\tan\vartheta)^{(0)} & 0 & -s \end{vmatrix} \qquad (2.64)$$

and its solution yields the characteristic polynomial

$$M(s)=s^4+A_1s^3+A_2s^2+A_3s+A_4, \tag{2.65}$$

where

$$\begin{aligned}
A_1 &= b'_{24}(\tan\vartheta)^{(0)}(\sin\alpha)^{(0)} - b'_{24}\left(\frac{\cos\theta}{\cos\vartheta}\right)^{(0)} + b_{34}(\cos\theta)^{(0)} - b_{11} + \\
&\quad + b'_{34}(\cos\theta)^{(0)} - b_{22};\\
A_2 &= -b'_{24}b_{35}(\tan\vartheta\cos\theta)^{(0)} - b_{12}b'_{24}(\sin\alpha)^{(0)} + b_{14}(\sin\alpha)^{(0)} - \\
&\quad - b_{11}b'_{24}(\tan\vartheta\sin\alpha)^{(0)} + b_{24}(\tan\vartheta\sin\alpha)^{(0)} - b_{24}\left(\frac{\cos\theta}{\cos\vartheta}\right)^{(0)} + \\
&\quad + b_{11}b'_{24}\left(\frac{\cos\theta}{\cos\vartheta}\right)^{(0)} - b_{11}b_{34}(\cos\theta)^{(0)} + (-b_{22}b_{24} - \\
&\quad - b_{11}b'_{34} - b_{22}b'_{34})(\cos\theta)^{(0)} + b_{11}b_{12} - b_{12}b_{21};\\
A_3 &= (b_{12}b'_{24}b_{35} + b_{14}b_{35} + b_{11}b_{12}b_{34} - b_{12}b_{21}b_{35} + \\
&\quad + b_{11}b_{12}b'_{34} - b_{12}b_{21}b'_{34})(\cos\theta)^{(0)} + b_{11}b'_{24}b_{35}(\tan\vartheta\cos\theta)^{(0)} - \\
&\quad - b_{24}b_{35}(\tan\vartheta\cos\theta)^{(0)} - b_{12}b_{24}(\sin\alpha)^{(0)} + b_{14}b_{22}(\sin\alpha)^{(0)} - \\
&\quad - b_{11}b_{24}(\tan\vartheta\sin\alpha)^{(0)} + b_{14}b_{21}(\tan\vartheta\sin\alpha)^{(0)} - \\
&\quad - b_{14}b_{21}\left(\frac{\cos\theta}{\cos\vartheta}\right)^{(0)} + b_{11}b_{24}\left(\frac{\cos\theta}{\cos\vartheta}\right)^{(0)};\\
A_4 &= (b_{12}b_{24}b_{35} - b_{14}b_{22}b_{35})(\cos\theta)^{(0)} + (b_{11}b_{24}b_{35} - \\
&\quad - b_{14}b_{21}b_{35})(\tan\vartheta\cos\theta)^{(0)}.
\end{aligned} \tag{2.66}$$

The characteristic polynomial M(s), equated to zero, yields the characteristic equation

$$M(s)=s^4+A_1s^3+A_2s^2+A_3s+A_4=0. \tag{2.67}$$

2.3.3. Free Motion

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Analysis of free lateral aircraft motion reduces to investigation of the roots of characteristic equation (2.67) (see Table 2.5). The sign of the real parts of the roots of equation (2.67) determines the stability (or instability) of unperturbed motion (2.62), and the form and the magnitude of the roots of characteristic equation (2.67) determine the nature of lateral free perturbed motion.

The presence of one pair of complex conjugate roots and two real roots among the roots of characteristic equation M(s) = 0 is typical of lateral perturbation of aircraft motion. In this case the polynomial M(s) can be written as the following product of co-factors:

$$M(s)=A_4(T^2s^2+2T\xi s+1)(T_1s+1)(T_2s+1). \tag{2.68}$$

Two aperiodic motions correspond to the two real roots, and oscillating motion corresponds to the pair of complex ones. All three partial motions exist simultaneously and, superimposed on one another, form lateral perturbed motion. In this case one of the two real roots is generally large in absolute magnitude in comparison with the other, but the complex roots normally have intermediate moduli.

Motion, corresponding to the real root which is large in absolute value, is called *rolling motion*. It consists basically in a change in the roll angle and angular velocity of roll.

Motion corresponding to the real root which is small in absolute value, is called *spiral*. Ordinarily this "small" root is greater than zero, which leads to the corresponding instability called *spiral instability*.

Motion corresponding to the pair of complex roots is called *oscillation*.

Spiral and oscillating motions produce *yaw motion*.

Example 3. This example illustrates the roots of characteristic equation (2.67). Flight of an aircraft at an elevation of 12,000 m at velocity 250m/sec is examined. Characteristic equation (2.67) has the form

$$s^4+1{,}909\,s^3+2{,}69\,s^2+3{,}95\,s-0{,}00437=0,$$

and the roots are:

$$s_1=-1{,}695;\ s_2=0{,}001105;\ s_{3,4}=-0{,}107\pm j\,1{,}525.$$

For aerodynamically axially symmetric aircraft, stabilized in roll or angular velocity of roll, the effect of roll motion on yaw can be disregarded. In this case the system of linearized equations for lateral perturbed motion (2.60)-(2.61) is broken down into two separate subsystems — a system of equations for perturbed yaw motion (2.69) and system of equations for perturbed roll (2.71). /135

2.3.4. Equations of Yaw

Linearized equations of perturbed isolated yaw are written in the form

$$\begin{aligned}&\frac{d^2\Delta\psi}{dt^2}-b_{22}\frac{d\Delta\psi}{dt}-b_{24}\Delta\beta-b'_{24}\frac{d\Delta\beta}{dt}=\\&=b_{27}\Delta\delta_{\mathrm{d}}+b'_{27}\frac{d\Delta\delta_{\mathrm{d}}}{dt}+b_{29}M_{y\,\mathrm{h}};\\&\frac{d\Delta\psi_{\mathrm{v}}}{dt}-b_{34}\Delta\beta=b_{37}\Delta\delta_{\mathrm{d}}+b_{39}Z_{\mathrm{h}},\end{aligned} \tag{2.69}$$

where $\Delta\psi - \Delta\beta - \Delta\psi_V = 0.$

In system (2.69) the unknowns, i.e., the output variables, are deviations $\Delta\psi$, $\Delta\beta$ and $\Delta\psi_V$; the input variables as known time functions are the control action $\Delta\delta_d$ and the perturbations M_{yh} and Z_h.

The zero solution of system (2.69):

$$\Delta\psi = \Delta\psi^{(0)} = 0; \quad \Delta\beta = \Delta\beta^{(0)} = 0; \quad \Delta\psi_V = \Delta\psi_V^{(0)} = 0, \tag{2.70}$$

which holds for $\Delta\beta_d = 0$; $M_{yh} = 0$; $Z_h = 0$, determines unperturbed yaw (see Table 2.1, item 25).

In equations (2.69) the *dynamic coefficients* b_{ij} are known time functions which can be considered constant within limits of the applicability of the frozen coefficients principle.

2.3.5. Transfer Functions and Frequency Responses of Aircraft in Yaw

Using system (2.69) within limits of the accuracy of the principle of "frozen coefficients," it is possible to find expressions for the transfer functions of dynamic axisymmetric aircraft (see Subsection 2.14), which characterize yaw motion.

Equations of yaw are no different from equations of short-period longitudinal perturbed motion. The equations given have identical form and the course angle ψ corresponds to the pitch angle ϑ, the course velocity angle ψ_V to the angle of inclination, the angle of skid β to the attack angle α, the angle of deflection of the yaw control elements $\Delta\delta_d$ to the angle of deflection of the pitch control elements $\Delta\delta_h$, and the dynamic coefficients b_{ij} to the dynamic coefficients a_{ij}.

By virtue of the indicated correspondence, transfer functions which characterize yaw motion do not differ in any way from transfer functions which describe short-period longitudinal perturbed motion. Further simplifications, analysis of free motion and transient processes, and also frequency responses which correspond to the indicated transfer functions also remain valid. /136

2.3.6. Equations of Roll Motion

The linearized equation for perturbed isolated roll motion is given as

$$\frac{d^2\Delta\gamma}{dt^2} - b_{11}\frac{d\Delta\gamma}{dt} = b_{18}\Delta\delta_e + b_{19}M_{x\,h}, \tag{2.71}$$

where the dynamic coefficients b_{ij} are:

$$b_{11}=\frac{M_x^{\omega_x}}{J_x};\quad b_{18}=\frac{M_x^{\delta_e}}{J_x};\quad b_{19}=\frac{1}{J_x}. \tag{2.72}$$

In equation (2.71) the output variables are the deviations $\Delta\gamma$ and $\Delta\dot{\gamma}$; the input variables as known time functions are the control action $\Delta\delta_a$ and the perturbation M_{xh}, where M_{xh} is the perturbing moment of roll, which can, in particular, account the effect of yaw motion on roll motion:

$$M_{xh}=\Delta M_{xh}+M_x^{\beta}\Delta\beta+M_x^{\delta_d}\Delta\delta_d+M_x^{\omega_y}\Delta\omega_y. \tag{2.73}$$

As during analysis of longitudinal perturbed motion, we may assume

$$M_{xh}=M_x^{\delta_e}\delta_e. \tag{2.74}$$

The zero solution of equation (2.71):

$$\Delta\gamma=\Delta\gamma^{(0)}=0;\quad \Delta\dot{\gamma}=\Delta\dot{\gamma}^{(0)}=0, \tag{2.75}$$

which is valid when $\Delta\delta_a = 0$, $M_{xh} = 0$, determines the unperturbed roll motion (see Table 2.1, item 25).

In equation (2.71) the dynamic coefficients, b_{ij} are known time functions. Inasmuch as the processes of stabilization of roll angle or roll angular velocity occur quite quickly, it is customary during investigation of these processes to "freeze" dynamic coefficients b_{ij}.

2.3.7. Transfer Functions and Frequency Responses of Aircraft in Roll /137

According to equation (2.71) within the limits of accuracy of the "frozen coefficients" principle we may write the following transfer functions:

$$Y_{\delta_e}^{\gamma}(s)=\frac{\Delta\gamma(s)}{\Delta\delta_e(s)}=\frac{K_e}{s(T_e s+1)}; \tag{2.76}$$

$$Y_{\delta}^{\dot{\gamma}}(s)=\frac{\Delta\dot{\gamma}(s)}{\Delta\delta(s)}=\frac{K_e}{T_e s+1}, \tag{2.77}$$

where K_a is the transfer coefficient of aircraft in roll:

$$K_e = -\frac{b_{18}}{b_{11}}, \tag{2.78}$$

and T_a is the time constant in roll motion:

$$T_e = -\frac{1}{b_{11}} = -\frac{I_x}{M_x^{\omega_x}}. \tag{2.79}$$

The time constant T_a, which characterizes the delay of aircraft reaction to deflection of roll control elements, is always positive because $M_x^{\omega_x} < 0$ always.

Examples of logarithmic frequency responses which correspond to transfer functions (2.76) and (2.77) are given in Figures 2.34-2.36.

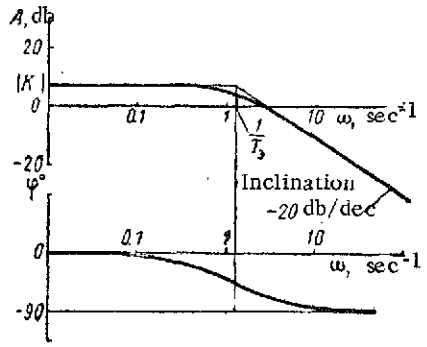

Figure 2.34. Logarithmic Frequency Responses of Transfer Function $\dot{\gamma}(s)/\delta_a(s)$; $|K_a| = 3$; $1/T_a = 1.2$ sec^{-1}.

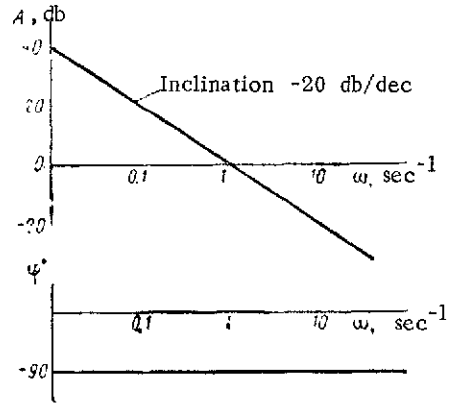

Figure 2.35. Logarithmic Frequency Responses of the Factor 1/s.

2.3.8. Transient Processes in Terms of Roll Angle and Angular Roll Velocity with Gradual Deflection of Roll Control Elements

The change in time of the angle $\Delta\gamma$ and angular velocity $\Delta\dot{\gamma}$ during gradual deflection of roll control elements is described by the following equations, which correspond to transfer functions (2.76) and (2.77):

$$\frac{\Delta\gamma}{\Delta\delta_e} = K_e\left[t - T_e\left(1 - e^{-\frac{t}{T_e}}\right)\right]; \tag{2.80}$$

$$\frac{\Delta\dot{\gamma}}{\Delta\delta_e} = K_e\left(1 - e^{-\frac{t}{T_e}}\right). \tag{2.81}$$

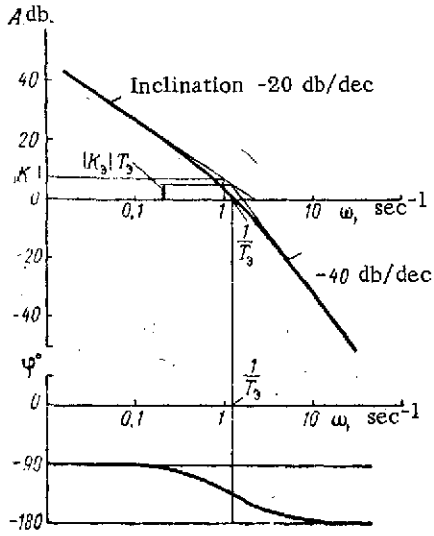

Figure 2.36. Logarithmic Frequency Responses of Transfer Function $\gamma(s)/\delta_a(s)$; $|K_a| = 3$; $1/T_a = 1.2\ \text{sec}^{-1}$.

An example of a transient process in terms of $\Delta\dot{\gamma}$, described by the relation (2.81), is illustrated in Figure 2.37.

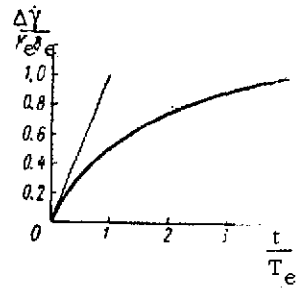

Figure 2.37. Nature of Transient Process $\Delta\dot{\gamma}(t)$ During Deflection of Roll Control Elements.

§2.4. Aircraft as a Nonlinear Controlled Object

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2.4.1. Equations of Motion

Perturbed motion of aircraft (see Section 2.1.7) is described in the general case by a system of non-linear differential equations in the form

$$\Delta\dot{x}_i = X_i(x_1^{(0)}, \ldots, x_n^{(0)};\ \Delta x_1, \ldots, \Delta x_n)\quad (i = 1, \ldots, n), \tag{2.82}$$

where n is the order of the system; $x_1^{(0)}, \ldots, x_n^{(0)}$ are variables which characterize unperturbed aircraft motion (for example, $V^{(0)}$, $\alpha^{(0)}$, $\dot{\vartheta}^{(0)}$, $\gamma^{(0)}$, etc.); $\Delta x_1, \ldots, \Delta x_n$ are variables which characterize perturbed aircraft motion (for example, ΔV, $\Delta\alpha$, $\Delta\dot{\vartheta}$, $\Delta\gamma$, etc.).

Assuming that functions of X_i permit expansion by powers of perturbations Δx_i into series, which converge at some sufficiently small M-vicinity of unperturbed motion

$$\sum_{i=1}^{n} \Delta x_i^2 \leqslant M, \tag{2.83}$$

and by performing such expansion, we can present system (2.82) in the form

$$\Delta\dot{x}_i=\sum_{k=1}^{n} a_{ik}(x_1^{(0)},\ldots,x_n^{(0)})\cdot\Delta x_k+\Delta X_i(x_1^{(0)},\ldots,x_n^{(0)},\ \Delta x_1,\ldots,\Delta x_n),\quad (i=1,\ldots,n), \tag{2.84}$$

where a_{ik} are the coefficients of the linear part of the expansion, and the functions ΔX_i contain only terms of second and higher order relative to perturbations Δx_1, ..., Δx_n.

Based on the smallness of perturbations Δx_i (i = 1, ..., n), we can disregard terms of second and higher orders relative to perturbations and assume that perturbed aircraft motion is described by a system of linear equations:

$$\Delta\dot{x}_i=\sum_{k=1}^{n} a_{ik}(x_1^0,\ldots,x_n^0)\Delta x_k\quad (i=1,\ldots,n). \tag{2.85}$$

Equations (2.85) are called *equations of first approximation*, or *linear approximation* of system (2.84).

Inasmuch as coefficients a_{ik} in system (2.85) are explicit functions of the parameters of *undisturbed* aircraft motion, which is assumed to be known, coefficients a_{ik} are, in the general case, known (given) time functions. Later on (as in Section 2.1-2.3) the "freezing principle" is assumed to be accurate, i.e., it is assumed that a_{ik} = const (i, k = 1, ..., n). If the condition of invariance of coefficient a_{ik} is not satisfied this fact is stipulated (see Section 2.4.4-2.4.6.). /140

In solving the problem of aircraft stability using system (2.85) much importance is attached to roots s_1, ..., s_n of the characteristic equation D(s) = 0 (see Section 2.1.4):

$$D(s)=|a_{ik}-s\delta_{ik}|=0\quad (i,\ k=1,\ldots,n),$$
$$a_{ik}=\text{const};\quad \delta_{ik}=1\quad i=k\ \text{и}\ \delta_{ik}=0\quad i\neq k. \tag{2.86}$$

where a_{ik} = const; δ_{ik} = 1 when i = k and δ_{ik} = 0 when i ≠ k.

Depending on the properties of the aircraft the system of linear equations (2.85) can, in turn, be divided into several independent subsystems, which describe, respectively, longitudinal perturbed motion (equation system (2.39), Section 2.2.1.), perturbed yaw (equation system (2.69), Section 2.3.4.) and perturbed roll (equation system (2.71), Section 2.3.).

In connection with the fact that the validity of substituting non-linear equations (2.84) by linear (2.85) is not obvious, the following problems arise:

a) finding conditions for which it is permissible to examine linear equations of first approximation (2.85) for investigating stability and the nature of perturbed aircraft motion;

b) designation of methods and procedures for investigating stability and the nature of aircraft motions and conditions when equations of first approximation (2.85) are unsuitable for these purposes.

2.4.2. The Admissibility of Analyzing Stability of Unperturbed Aircraft Motion by Equations of First Approximation

The discussion is limited to perturbations in the M-vicinity (2.83) of unperturbed steady motion, that is to quite small perturbations (Δx_1, ..., Δx_n, the admissibility of studying aircraft stability by linear equations (2.85) instead of non-linear (2.84) is determined by the following theorems of A. M. Lyapunov.

Theorem 1. If the real parts of all roots s_1, ..., s_n of characteristic equation (2.86) of the first approximation are negative, unperturbed motion ($x_1^{(0)}$, ..., $x_n^{(0)}$) is asymptotically stable, regardless of the non-linear terms ΔX_i ($i = 1, \ldots, n$) of equations (2.84).

Theorem 2. If at least one root with a positive real part exists among roots s_1, ..., s_n of characteristic equation (2.86) of the first approximation, then perturbed motion $x_1^{(0)}$, ..., $x_n^{(0)}$) is unstable, regardless of the non-linear terms ΔX_1 ($i = 1, \ldots, n$) of equations (2.84).

Cases when there are zero roots or roots with zero real parts among the roots of characteristic equation (2.86) are called *critical cases*. In all critical cases, as Lyapunov showed, the problem of stability of unperturbed motion cannot be solved with the aid of first approximation equations (2.85), but must be solved with consideration of non-linear functions ΔX_i ($i = 1, \ldots, n$), i.e., equations of perturbed motion in complete form (2.84) or (2.82). /141

In this manner, the absence of zero or purely imaginary roots of characteristic equation (2.86) is the general indication of the admissibility of using first approximation equations for analyzing aircraft motion stability for quite small perturbations Δx_1, ..., Δx_n.

Example 4. It is assumed that longitudinal perturbed short-period aircraft motion (see Section 2.2.2 and 2.2.3) is described by the non-linear differential equation

$$\Delta\ddot{\alpha} + \Delta\alpha = A\Delta\dot{\alpha}(3\Delta\alpha^2 - \Delta\dot{\alpha}^2),$$

in which the form of the right-hand side is governed by the non-linearity of the aerodynamic characteristics of the aircraft.

In the symbols $\Delta\alpha = \Delta x_1$, $\Delta\dot{\alpha} = \Delta x_2$ the initial equation is equivalent to the following equation system:

$$\left.\begin{aligned} \Delta\dot{x}_1 &= -\Delta x_2 + A\Delta x_1^3; \\ \Delta\dot{x}_2 &= \Delta x_1 + A\Delta\dot{x}_2^3. \end{aligned}\right\} \qquad (1)$$

for which the characteristic equation of first approximation $s^2 + 1 = 0$ has two purely imaginary roots: $s_{1.2} = \pm j$. The corresponding solution based on first approximation of initial perturbations Δx_{10}, Δx_{20} acquires the form

$$\begin{aligned} \Delta x_1 &= \Delta x_{10}\cos t - \Delta x_{20}\sin t; \\ \Delta x_2 &= \Delta x_{10}\sin t + \Delta x_{20}\cos t. \end{aligned}$$

By definition of stability (see Section 2.1.7) the state of aircraft equilibrium $\Delta x_1 = 0$, $\Delta x_2 = 0$ is stable (but non-asymptotically!), because for any given integer $\varepsilon > 0$ on the condition

$$|\Delta x_{10}| \leqslant \frac{\varepsilon}{2}; \quad |\Delta x_{20}| \leqslant \frac{\varepsilon}{2}$$

the inequalities

$$|\Delta x_1| \leqslant \varepsilon; \quad |\Delta x_2| \leqslant \varepsilon$$

are satisfied for all $t \geqslant 0$. However, the given case relates to the class of critical cases and therefore the problem of stability may be exactly resolved only in consideration of non-linear terms in equations of perturbed motion.

Using Lyapunov's direct method (see Section 2.4.3), the Lyapunov function for this purpose can be written as

$$V(\Delta x_1, \Delta x_2) = \frac{1}{2}\left(\Delta x_1^2 + \Delta x_2^2\right),$$

and the derivative dV/dt, computed in system (I), is

$$\frac{dV}{dt} = A\left(\Delta x_1^4 + \Delta x_2^4\right).$$

According to theorems in Section 2.4.3 unperturbed aircraft motion which corresponds to the condition of equilibrium $\Delta x_1 = 0$, $\Delta x_2 = 0$, is stable asymptotically for any initial perturbation Δx_{10}, Δx_{20} when $A < 0$ and unstable when $A > 0$. The case $A = 0$ reduces the problem to a linear one. /142

2.4.3. Lyapunov's Direct Method for Analyzing Stability of Steady-State Motion of Aircraft

Lyapunov's direct method for solving the problem of motion stability is successfully applied in cases when first approximation equations (2.85) are unsuitable for this purpose and when it is necessary to use perturbed motion equations in the complete form

$$\Delta\dot{x}_i = X_i(\Delta x_1, \ldots, \Delta x_n) \quad (i = 1, \ldots, n), \tag{2.87}$$

where all of the symbols and assumptions correspond to system (2.82). It is assumed that the functions X_i do not depend explicitly on time t, i.e., steady unperturbed aircraft motion is studied.

Lyapunov's direct method amounts to construction of such functions V of variables $\Delta x_1, \ldots, \Delta x_n$, whose complete derivatives according to equations (2.87) have certain specific properties.

It is assumed relative to V-functions, that they are unambiguous, are defined in some sufficiently small H-vicinity of unperturbed motion

$$\sum_{i=1}^{n} \Delta x_i^2 < H, \tag{2.88}$$

vanish when $\Delta x_1 = \ldots = \Delta x_n = 0$ and have continuous partial derivatives.

The function $V(\Delta x_1, \ldots, \Delta x_n)$ is called *sign-fixed* (positive-fixed or negative-fixed), if in range (2.88) it acquires only one specific sign and vanishes only when $\Delta x_1 = \ldots = \Delta x_n = 0$.

The function $V(\Delta x_1, \ldots, \Delta x_n)$ is called *sign-constant* (positive or negative), if in range (2.88) it acquires only one specific sign, but can also vanish when $\Delta x_1^2 + \ldots + \Delta x_n^2 \neq 0$.

The function $V(\Delta x_1, \ldots, \Delta x_n)$ is called *sign-variable* if it is neither sign fixed nor sign-constant and, consequently, no matter what the number H is, can acquire both positive and negative values in range (2.88).

Example 5. For n = 3 the functions

$$V = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2; \qquad V = \Delta x_1^2 + 2\Delta x_1 \Delta x_2 + 2\Delta x_2^2 + \Delta x$$

are fixed as positive no matter how great the magnitude of H in (2.88). /143

The function $V = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = \Delta x_3^3$ is also positive-fixed, but the magnitude of H in (2.88) must be assumed sufficiently small.

The functions $V = \Delta x_1^2 + \Delta x_2^2 + 2\Delta x_1 \Delta x_2 + \Delta x_3^2$; $V = \Delta x_1^2 + \Delta x_2^2$ are sign-constant (positive), the functions $V = \Delta x_1$; $V = \Delta x_1^2 + \Delta x_2^2 - \Delta x_3^4$ are sign-variable.

If V is a sign-fixed function, the equation V = C = const represents a single-parameter group of closed surfaces. When parameter C decreases each surface contracts to the coordinate origin $\Delta x_1 = \ldots = \Delta x_n = 0$, but at the limit $C \to 0$ it diminishes to a point — the coordinate origin. The surfaces intersect all the rays from the coordinate origin to infinity.

In addition to the functions $V(\Delta x_1, \ldots, \Delta x_n)$ their complete derivatives in time are also examined according to equations (2.87):

$$\dot{V} = \frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial \Delta x_i} \frac{d\Delta x_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial \Delta x_i} X_i = W(\Delta x_1, \ldots, \Delta x_n). \tag{2.89}$$

Consequently, the function $\dot{V} = W$ is also a function of the variables Δx_1, $\ldots$, Δx_n, which vanish when $\Delta x_1 = \ldots = \Delta x_n = 0$.

The essence of Lyapunov's direct method for steady motions consists in the following basic theorems.

Lyapunov's first theorem of stability. If it is possible to find a sign-fixed function $V(\Delta x_1, \ldots, \Delta x_n)$ for differential equations of perturbed motion (2.87), whose complete derivative in time $\dot{V}$, compiled by virtue of these equations (2.89), is a sign-constant function with sign opposite that of V, or identically equal to zero, unperturbed motion is stable.

Lyapunov's second theorem of stability. If it is possible to find a sign-fixed function $V(\Delta x_1, \ldots, \Delta x_n)$ for differential equations of perturbed motion (2.87), whose complete derivative in time $\dot{V}$, compiled by virtue of these equations, is also a sign-fixed function with sign opposite that of V, then unperturbed motion is asymptotically stable.

Lyapunov's theorem of instability. If it is possible to find a function $V(\Delta x_1, \ldots, \Delta x_n)$ for differential equations of perturbed motion (2.87) such /144
that its complete derivative in time $\dot{V}$, compiled by virtue of these equations

(see (2.89)), is a sign-fixed function, but the function V itself is not sign-constant with sign opposite that of $\dot{V}$, then unperturbed motion is unstable.

M. G. Chetayev's theorem of instability. If it is possible to find a function $V(\Delta x_1, \ldots, \Delta x_n)$ for differential equations of perturbed motion (2.87) such that a) a region exists in any size vicinity of the origin of the coordinates where $V > 0$ on the boundary of which $V = 0$, and b) in all points of region $V > 0$ derivative $\dot{V}$ (see (2.89)) acquires positive values, then unperturbed motion is unstable.

Functions which satisfy one of the above theorems are called *Lyapunov functions* or *Chetayev functions*.

V. V. Rumyantsev's theorem of stability in relation to some of the variables. When differential equations of perturbed motion are such that it is possible to find a sign-fixed function V in relation to the variables $\Delta x_1, \ldots, \Delta x_k$, whose derivative would be a sign constant function with sign opposite that of V by virtue of these equations or identically equal to zero, then unperturbed motion is stable in relation to the variables $\Delta x_1, \ldots, \Delta x_k$.

Note. When the number of values Δx_i, in relation to which stability is investigated, is equal to n, V. V. Rumyantsev's theorem is equivalent to Lyapunov's first theorem of stability.

For practical application of Lyapunov's direct method it is necessary to know methods for constructing Lyapunov functions, and also the criteria of sign-fixedness and sign-variability of the functions V. The simplest methods for constructing Laypunov functions are given in Sections 2.4.4 and in examples of Section 2.4. Because there are no general criteria for sign-fixedness and sign-variability of functions V, certain simple criteria are given below, with which this problem is easily solved in many special cases.

Certain Features of Sign-Fixedness and Sign-Variability of Functions

It is assumed that the function $V(\Delta x_1, \ldots, \Delta x_n)$ is homogeneous of arbitrary m-th order. The arbitrary function $W(\Delta x_1, \ldots, \Delta x_n)$, which vanishes when $\Delta x_1 = \ldots = \Delta x_n = 0$ and which satisfies the inequality in region (2.88).

$$|W(\Delta x_1, \ldots, \Delta x_n)| < A\{|\Delta x_1| + \ldots + |\Delta x_n|\}^m, \qquad (2.90)$$

where A is some constant, is also examined.

In addition to representing the V-function as a form, we examine an arbitrary V-function which expands into a series by powers of $\Delta x_1, \ldots, \Delta x_n$ in a certain

vicinity of the coordinate origin $\Delta x_1 = \ldots = \Delta x_n = 0$. It is assumed that this expansion begins with terms of some arbitrary order m, such that the following notation is valid: /145

$$V(\Delta x_1, \ldots, \Delta x_n) = V_m(\Delta x_1, \ldots, \Delta x_n) + V^*(\Delta x_1, \ldots, \Delta x_n), \tag{2.91}$$

Here V_m is a form of the m-th order, and V* is the set of terms of higher orders relative to $\Delta x_1, \ldots, \Delta x_n$. Here the function $V^*(\Delta x_1, \ldots, \Delta x_n)$ can be examined, and in this case by numerous procedures, like the form of m-th order, whose coefficients are functions of $\Delta x_1, \ldots, \Delta x_n$, which vanish when $\Delta x_1 = \ldots = \Delta x_n = 0$. Consequently, when H, which determines region (2.88), is sufficiently small the indicated coefficients will also be as small as desired.

Given these assumptions and conditions the following statements are valid.

1. Any form V of an odd order m is a sign-variable function. But if m is an even number the V may be either sign-fixed or sign-variable. The problem as to which of these cases actually occurs is very complex for forms of order higher than the second if the number of independent variables is more than two. For second-order forms (with any number of independent variables) this problem is resolved simply with the aid of the following criterion, known as Silvester's theorem.

2. Silvester's Theorem. In order that the quadratic form

$$2V = \sum_{\alpha, \beta = 1}^{n} C_{\alpha\beta} \Delta x_\alpha \Delta x_\beta \tag{2.92}$$

be positive-fixed, it is necessary and sufficient that all angular minors of its discriminant, i.e., the values

$$C_{11}, \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}, \ldots, \begin{vmatrix} C_{11} & C_{12} \ldots C_{1n} \\ C_{12} & C_{22} \ldots C_{2n} \\ \ldots\ldots\ldots \\ C_{1n} & C_{2n} \ldots C_{nn} \end{vmatrix}$$

be positive.

3. When V is a sign-fixed form of m-th order, the function

$$U(\Delta x_1, \ldots, \Delta x_n) = V(\Delta x_1, \ldots, \Delta x_n) + W(\Delta x_1, \ldots, \Delta x_n) \tag{2.93}$$

will also be sign-fixed with the same sign for any choice of function $W(\Delta x_1, \ldots, \Delta x_n)$, which satisfies inequality (2.90) in region (2.88), where A is a sufficiently small positive number, which depends exclusively on coefficients of form V. When V is a sign-variable form the function U will also be sign-variable for the same conditions. /146

4. The sign-fixedness or sign-variability of a configuration is preserved if any form of the same order, having sufficiently small coefficients, is added to it.

The validity of this assumption follows directly from the fact that any form of m-th order necessarily satisfies inequality (2.90), while the coefficient A can be small as desired when coefficients of the form are sufficiently small.

5. If $V_m(\Delta x_1, \ldots, \Delta x_n)$ is a sign-fixed form, function (2.91) will be sign-fixed; if $V_m(\Delta x_1, \ldots, \Delta x_n)$ is a sign-variable form function (2.91) will be sign-variable.

The latter assertion remains valid if the assumption that the function V* in expression (2.91) is analytical with expansion beginning with terms of order no lower than (m + 1), is replaced by the more general assumption that V* vanishes when $\Delta x_1 = \ldots = \Delta x_n = 0$ and has order of smallness higher than m, i.e., that V* satisfies the inequality

$$|V^*(\Delta x_1, \ldots, \Delta x_n)| < A\{|\Delta x_1| + \ldots + |\Delta x_n|\}^{m+\alpha}, \qquad (2.94)$$

in some vicinity of the coordinate origin, where α is a positive number which may be as small as desired. Function V* need not be analytical in this case.

Thus, sign fixedness and sign-variability of analytical functions are determined by the set of members of lowest-order in the expansions of these functions with the exception of the case when this set of members of least order represents a sign-constant form. However, when the function is not a form, its sign-fixedness or sign-variability can be upset by the addition of terms of higher order.

Example 6. The example pertains to the use of Lyapunov's direct method for investigating stability of relative equilibrium of a spacecraft (SC) in motion relative to its center of mass in the Earth's gravitational field [18]. For constructing the Lyapunov function properties of the first integral are used.

SC motion is examined in the following assumptions:

1) the gravitational field is assumed to be central Newtonian;

2) SC is an absolutely solid body, whose dimensions are negligibly small in comparison with the orbit of motion of the center of mass of the SC;

3) the SC center of mass moves in a Keplerian circular orbit;

4) the motion studied near the center of mass does not affect the motion of the center of mass so that the orbit of the center of mass of the SC remains unperturbed, i.e., Keplerian circular;

5) only forces of gravity act on the SC, no other forces being applied.

With the given assumptions the equations of spacecraft motion relative to its center of mass in connected coordinate system $Ox_{11}y_{11}z_{11}$, whose axes are directed along the main central axes of inertia of the spacecraft, are written as Euler equations: /147

$$\left.\begin{aligned} &I_x\dot{\omega}_x + (I_z - I_y)\,\omega_y\omega_z = 3\Omega^2(I_z - I_y)\,\gamma'\gamma'';\\ &I_y\dot{\omega}_y + (I_x - I_z)\,\omega_z\omega_x = 3\Omega^2(I_x - I_z)\,\gamma''\gamma;\\ &I_z\dot{\omega}_z + (I_y - I_x)\,\omega_x\omega_y = 3\Omega^2(I_y - I_x)\,\gamma\gamma', \end{aligned}\right\} \qquad (I)$$

where Ω is the constant angular velocity of the spacecraft's center of mass in orbit; ω_x, ω_y, ω_z are components of angular velocity in relative motion.

For relative direction cosines α, α', α''; β, β', β''; γ, γ', γ'', which determine the mutual positions of connected $Ox_{11}y_{11}z_{11}$ in the orbital Oxyz coordinate systems with the aid of the matrix

	x_{11}	y_{11}	z_{11}
x	α	α'	α''
y	β	β'	β''
z	γ	γ'	γ''

the following kinematic relations are valid:

$$\left.\begin{aligned} &\dot{\gamma} = \gamma'\omega_z - \gamma''\omega_y + \alpha\Omega; \qquad \dot{\alpha} = \alpha'\omega_z - \alpha''\omega_y - \gamma\Omega;\\ &\dot{\gamma}' = \gamma''\omega_x - \gamma\omega_z + \alpha'\Omega; \qquad \dot{\alpha}' = \alpha''\omega_x - \alpha\omega_z - \gamma'\Omega;\\ &\dot{\gamma}'' = \gamma\omega_y - \gamma'\omega_x + \alpha''\Omega; \qquad \dot{\alpha}'' = \alpha\omega_y - \alpha'\omega_x - \gamma''\Omega;\\ &\dot{\beta} = \beta'\omega_z - \beta''\omega_y; \quad \dot{\beta}' = \beta''\omega_x - \beta\omega_z; \quad \dot{\beta}'' = \beta\omega_y - \beta'\omega_x. \end{aligned}\right\} \qquad (II)$$

Equation system (I)-(II) is closed because the matrix of the direction cosines has the properties:

1) the sum of squares of the elements of a row (column) is equal to unity; for example, $\alpha^2 + \beta^2 + \gamma^2 = 1$;

2) the sum of paired products of elements of two rows (columns) is zero; for example, $\alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0$;

3) each element of the matrix is equal to its algebraic complement; for example, $\alpha' = \beta''\gamma - \beta\gamma''$.

System (I)-(II) is difficult to integrate in finite form; the first integral exists for it for arbitrary moments of inertia I_x, I_y, I_z:

$$V = \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) + \frac{3}{2}\Omega^2\left(I_x\gamma^2 + I_y\gamma'^2 + I_z\gamma''^2\right) - \Omega\left(I_x\omega_x\beta + I_y\omega_y\beta' + I_z\omega_z\beta''\right) = \text{const},$$

whose expression through relative angular velocities

$$\bar{\omega}_x = \omega_x - \Omega\beta; \quad \bar{\omega}_y = \omega_y - \Omega\beta'; \quad \bar{\omega}_z = \omega_z - \Omega\beta''$$

acquires the form

$$V = \frac{1}{2}\left(I_x\bar{\omega}_x^2 + I_y\bar{\omega}_y^2 + I_z\bar{\omega}_z^2\right) + \frac{3}{2}\Omega^2\left(I_x\gamma^2 + I_y\gamma'^2 + I_z\gamma''^2\right) - \frac{1}{2}\Omega^2\left(I_x\beta^2 + I_y\beta'^2 + I_z\beta''^2\right) = \text{const}.$$

The partial solution of equations of motion which determines the relative equilibrium of a spacecraft in a circular orbit is viewed as the unperturbed motion whose stability is studied:

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$$\bar{\omega}_x^{(0)} = \bar{\omega}_y^{(0)} = \bar{\omega}_z^{(0)} = 0; \qquad \alpha^{(0)} = \beta'^{(0)} = \gamma''^{(0)} = 1;$$
$$\alpha'^{(0)} = \alpha''^{(0)} = \beta^{(0)} = \beta''^{(0)} = \gamma^{(0)} = \gamma'^{(0)} = 0.$$

For such direction cosines the y_{11} axis coincides with the normal to the plane of the orbit, z_{11} axis coincides with the radius-vector, the x_{11} axis coincides with the tangent to the circular orbit; The components of angular velocity and relative equilibrium of the spacecraft are $\omega_x = \omega_z = 0$; $\omega_y = \Omega$, i.e., one side of the spacecraft in relative equilibrium is always "looking" at the Earth.

Any motion, perturbed or unperturbed, can be presented in the form

$$\bar{\omega}_x = \Delta\bar{\omega}_x; \quad \bar{\omega}_y = \Delta\bar{\omega}_y; \quad \bar{\omega}_z = \Delta\bar{\omega}_z;$$
$$\alpha = 1 + \Delta\alpha; \quad \alpha' = \Delta\alpha'; \quad \alpha'' = \Delta\alpha'';$$
$$\beta = \Delta\beta; \qquad \beta' = 1 + \Delta\beta'; \quad \beta'' = \Delta\beta'';$$
$$\gamma = \Delta\gamma; \quad \gamma' = \Delta\gamma'; \quad \gamma'' = 1 + \Delta\gamma''.$$

where the index "Δ" denotes perturbed spacecraft motion.

The integral V in the variables of perturbed motion is written in the form

$$V = \frac{1}{2}\left(I_x\Delta\bar{\omega}_x^2 + I_y\Delta\bar{\omega}_y^2 + I_z\Delta\bar{\omega}_z^2\right) + \frac{3}{2}\Omega^2\left[(I_x + I_z)\Delta\gamma^2 + \right.$$
$$\left. + (I_y - I_z)\Delta\gamma'^2\right] + \frac{1}{9}\Omega^2\left[(I_y - I_x)\Delta\beta^2 + (I_y - I_z)\Delta\beta'^2\right].$$

with consideration of the first property of the matrix of the direction cosines.

The function V when $I_y > I_x > I_z$ is a positive fixed function because it is positive for any perturbed motion, but vanishes only during unperturbed motion. The derivative of V in terms of t is identically equal to zero by virtue of the equations of motion for determination of the first integral. Consequently, according to Lyapunov's theories of instability, the inequalities $I_y > I_x > I_z$ are sufficient conditions of stability of the examined relative equilibrium.

2.4.4. Lyapunov's Direct Method for Investigating Stability of Unsteady Aircraft Motion

Lyapunov's direct method for solving the problem of stability of unsteady aircraft motion is successfully used when equations of first approximation (2.85) are unsuitable for this purpose and it is necessary to use equations of perturbed aircraft motion in complete form:

$$\Delta\dot{x}_i = X_i(\Delta x_1, \ldots, \Delta x_n, t) \quad (i = 1, \ldots, n), \tag{2.95}$$

where the symbols and assumptions correspond to system (2.82), and the explicit time-dependence of the non-linear functions X_i is the consequence of the unsteady nature of unperturbed motion.

Just as in Section 2.4.3, the use of Lyapunov's direct method in the given case reduces to construction of such functions $V(\Delta x_1, \ldots, \Delta x_n, t)$, whose complete derivatives of t according to equations (2.95) have certain specific properties. In the given case the functions V, in contrast to Section 2.4.3, are explicit functions of t.

Relative to functions $V(\Delta x_1, \ldots, \Delta x_n, t)$, which are given in the region /149

$$\sum_{i=1}^{n}\Delta x_i^2 \leqslant H; \quad t \geqslant t_0 > 0, \tag{2.96}$$

where H is a sufficiently small constant, it is assumed that they have continuous partial derivatives in the given region for all variables and vanish when $\Delta x_1 = \ldots = \Delta x_n = 0$.

The function $V(\Delta x_1, \ldots, \Delta x_n, t)$ permits a *vanishingly small upper limit*, if it approaches zero when $\sum_{i=1} \Delta x_i^2 \to 0$ uniformly relative to t.

Example 7. The function $V = \sin t \sum_{i=1}^{n} \Delta x_i$ permits a vanishingly small upper limit, but the function $\sin \left[t \left(\sum_{i=1}^{n} \Delta x_i \right) \right]$ does not permit such a limit.

The function $V(\Delta x_1, \ldots, \Delta x_n, t)$ is called *sign-constant positive* (or negative) if it cannot acquire negative (or positive) values in region (2.96).

Function $V(\Delta x_1, \ldots, \Delta x_n, t)$ is called *positive definite*, if it satisfies the inequality

$$V(\Delta x_1, \ldots, \Delta x_n, t) \geqslant W(\Delta x_1, \ldots, \Delta x_n), \tag{2.97}$$

in region (2.96), but function $V(\Delta x_1, \ldots, \Delta x_n, t)$ is called *negative definite*, if it satisfies the inequality

$$V(\Delta x_1, \ldots, \Delta x_n, t) \leqslant -W(\Delta x_1, \ldots, \Delta x_n), \tag{2.98}$$

in region (2.96), where $W(\Delta x_1, \ldots, \Delta x_n)$ is a positive definite function which does not depend on t (see Section 2.4.3).

Example 8. The function $V_1 = (2 + \sin t) \sum_{i=1}^{n} \Delta x_i^2$ is positive definite because $V_1 \geqslant \sum_{i=1}^{n} \Delta x_i^2$; the function $V_2 = (-2 + \sin t) \sum_{i=1}^{n} \Delta x_i^2$ is negative definite, because $V_2 \leqslant -\sum_{i=1}^{n} \Delta x_i^2$; the function $V_3 = e^{-t} \left(\sum_{i=1}^{n} \Delta x_i^2 \right)$ is only sign-constant (positive), because it, vanishing when $\Delta x_1 = \ldots = \Delta x_n = 0$, approaches zero when $t \to \infty$ with

fixed values for $\Delta x_1, \ldots, \Delta x_n$ and, consequently, condition (2.97) is not satisfied.

In addition to the functions $V(\Delta x_1, \ldots, \Delta x_n, t)$ their complete time derivatives are also examined by virtue of equations (2.95):

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial \Delta x_i} X_i. \qquad (2.99)$$

The foundation of Lyapunov's direct method for unsteady motion consists of /150 the following basic theorems.

Lyapunov's first theorem of stability. If it is possible to find a sign-fixed function $V(\Delta x_1, \ldots, \Delta x_n, t)$ for differential equations of perturbed motion (2.95), for which the time derivative dV/dt by virtue of equations (2.95), for which the time derivative dV/dt by virtue of equations (2.95) is a sign-constant function with sign opposite that of V, or vanishes, unperturbed motion is stable.

Lyapunov's second theorem of stability. If the derivative dV/dt (see (2.99)) is sign-fixed when satisfying conditions of the preceding theorem, but the very function $V(\Delta x_1, \ldots, \Delta x_n, t)$ permits an infinitely small upper limit, unperturbed motion is asymptotically stable.

Lyapunov's theorem of instability. If a function $V(\Delta x_1, \ldots, \Delta x_n, t)$ exists which permits an infinitely small upper limit, whose time derivative dV/dt (see (2.99)) is a sign-fixed function, but the function V itself in region (2.96) may acquire values of the same sign as can dV/dt, unperturbed motion is unstable.

M. G. Chetayev's theorem of instability. If it is possible to find a function $V(\Delta x_1, \ldots, \Delta x_n, t)$ for differential equations of perturbed motion (2.95), which satisfies the conditions:

1) a region $V > 0$ exists in region (2.96);

2) the function V is bounded in region $V > 0$;

3) in region $V > 0$ the derivative dV/dt, compiled according to (2.99), acquires positive values and for all values $\Delta x_1, \ldots, \Delta x_n, t$, connected by relation

$$V(\Delta x_1, \ldots, \Delta x_n, t) \geqslant a,$$

where a is any positive integer, the inequality

$$dV/dt \geqslant l,$$

where l is also some positive integer which depends on α, unperturbed motion is unstable.

Functions which satisfy the given theorems are called *Lyapunov functions* or *Chetayev functions*.

In the case when equations of perturbed motion permit the existence of a sign-fixed first integral, this integral can be viewed as a Lyapunov function. The derivative of this function is identically equal to zero (according to the property of the first integral) and by virtue of the equations of motion and, according to Lyapunov's first theorem of stability, unperturbed motion will be stable in relation to the variables relative to which the first integral is a sign-fixed function.

In the case when equations of perturbed motion permit the existence of k first integrals /151

$$\Phi_l(\Delta x_1, \ldots, \Delta x_n; t) = \text{const} \quad (l = 1, \ldots, k), \tag{2.100}$$

each of which is not a sign-fixed function, according to the method suggested by N. G. Chetayev, the function

$$V = F(\Phi_1, \Phi_2, \ldots, \Phi_k, \lambda_1, \ldots, \lambda_m), \tag{2.101}$$

can be viewed as a V-function which depends on several undefined parameters $\lambda_1, \ldots, \lambda_m$, selected so that function V will be sign-fixed. In the simplest cases function V represents a linear combination (bundle) of first integrals Φ_i. When the resulting function is sign-fixed the unperturbed motion being studied will be stable because $\dot{V} \equiv 0$ according to the property of first integrals.

2.4.5. Stability of Aircraft Motion with Constantly (Continuously) Acting Perturbations

In the preceding sections aircraft motion stability was viewed in relation to instantaneous perturbations, i.e., initial deviations of $\Delta x_i(0)$ of aircraft motion parameters x_i from their values $x_i^{(0)}$ in unperturbed motion. However, in such form certain perturbations of aircraft motion cannot be completely considered and therefore the study of aircraft motion stability is of great practical importance in relation to perturbations which act constantly (continuously). In this case equations of perturbed motion for aircraft (2.82) are presented in the form

$$\Delta\dot{x}_i = X_i(\Delta x_1, \ldots, \Delta x_n, t) + Q_i(\Delta x_1, \ldots, \Delta x_n, t) \quad (i = 1, \ldots, n), \tag{2.102}$$

where functions Q_i correspond to contantly acting perturbations, for example forces or moments.

Relative to the right-hand side of equations (2.102) it is assumed that they are continuous in the region

$$\sum_{i=1}^{n} \Delta x_i^2 \leqslant H; \; t \geqslant 0 \tag{2.103}$$

that they permit the existence of a single solution for given initial conditions, that they satisfy the condition $X_i(0, \ldots, 0, t) = 0$, and that the practically unknown functions Q_i are sufficiently small in modulus for all variables Δx_i and t.

Unperturbed aircraft motion $(x_1^{(0)}, \ldots, x_n^{(0)})$ is called *stable in the presence of constantly acting perturbations*, if for any given positive integer A, no matter how small, two other positive integers $\lambda(A)$ and $\varepsilon(A)$ exist such that for all perturbations $\Delta x_i^{(0)}$ which satisfy the condition /152

$$\sum_{i=1}^{n} \Delta x_i^2(0) \leqslant \lambda, \tag{2.104}$$

and for any perturbing functions Q_i which satisfy the condition

$$\sum_{i=1}^{n} Q_i^2(t, \Delta x_1, \ldots, \Delta x_n) \leqslant \varepsilon, \tag{2.105}$$

any perturbed motion $(\Delta x_1, \ldots, \Delta x_n)$, defined by equations (2.102), satisfies the inequality

$$\sum_{i=1}^{n} \Delta x_i^2 < A \tag{2.106}$$

for any $t > 0$; otherwise unperturbed aircraft motion is called *unstable*.

In other words, unperturbed motion $(x_1^{(0)}, \ldots, x_n^{(0)})$ is stable in the presence of constantly acting perturbations if the values of Δx_i remain small

all the time, assuming they were small at the initial moment of time, and if perturbations Q_i are also small.

To analyze the stability of unperturbed aircraft motion with constantly acting perturbations it is possible to use Lyapunov's direct method (Section 2.4.4). In this case the following theorems of I. G. Malkin are valid.

Theorem 1. If there exists a positive definite function V for differential equations of perturbed motion (2.102), whose complete derivative of time, compiled from equations (2.102), is a negative definite function, and if the partial derivatives $\partial v/\partial \Delta x_i$ (i = 1, ..., n) are bounded in region (2.103), unperturbed motion is stable in the presence of constantly acting perturbations.

Theorem 2. If equations of perturbed motion are linear and have the form

$$\Delta \dot{x}_l = \sum_{k=1}^{n} a_{lk} \Delta x_k + f(t) \quad (l = 1, \ldots, n). \tag{2.107}$$

where f(t) is an arbitrary and limited function of t, then the conditions of Theorem 1 are not only sufficient but also necessary for stability with constantly acting perturbations.

Theorem 3. If equations of perturbed motion do not explicitly contain t, in order that unperturbed motion be stable with constantly acting perturbations, it is necessary and sufficient that it be stable asymptotically in the sense of Lyapunov, i.e., when $Q_i \equiv 0$ (i = 1, ..., n) in equations (2.102). /153

2.4.6. Aircraft Motion Stability in Finite Time Interval

For a number of problems the assumption of invariance of dynamic aircraft coefficients (see Sections 2.2.1, 2.3.1 and 2.3.3) or of the validity of the coefficient "freezing principle" are untrue or unverified. In such problems it must be considered that in equations of perturbed aircraft motion (see Section 2.4.1).

$$\Delta \dot{x}_l = \sum_{k=1}^{n} a_{lk}(t) \Delta x_k + \Delta X_l(\Delta x_1, \ldots, \Delta x_n, t) \quad (l = 1, \ldots, n) \tag{2.108}$$

the coefficients $a_{ik}(t)$ of the linear part of the expansion are *explicit* functions of time t. The functions ΔX_i are nonlinear functions of the variables $\Delta x_1, \ldots, \Delta x_n$ with coefficients which also depend on time t.

The explicit dependence of the right-side of equations (2.108) on time t is a consequence of the unsteady nature of unperturbed aircraft motion which changes, for example, according to a definite program.

Based on the notion that the interval of aircraft motion in time can be limited by a finite integer T, in the given case it is possible to state the problem of the stability of aircraft motion in a finite time interval. In this case it is assumed that the coefficients $a_{ik}(t)$, entering in equations (2.108) are real, limited and continuous functions of time t, and can be presented as the sums

$$a_{ik}(t)=a_{ik}(0)+\Delta a_{ik}(t) \quad (i, k=1,\ldots, n), \tag{2.109}$$

where $a_{ik}(0)$ are values of the coefficients a_{ik} at moment of time $t = t_0$, taken as the initial moment of time, and $\Delta a_{ik}(0) = 0$. The equation

$$D(s)=|a_{ik}(0)-s\delta_{ik}|=0 \quad (i, k=1,\ldots, n), \tag{2.110}$$

where $a_{ik}(0) = \text{const}$ ($\delta_{ik} = 1$ when $i = k$, $\delta_{ik} = 0$ when $i \neq k$), is a characteristic equation (see Section 2.13), which corresponds to equations of perturbed motion (2.108) for $t = t_0$.

Unperturbed aircraft motion is called *stable in a finite time interval* $[t_0, t_0 + \tau]$, when the variables $\Delta x_i(t)$, which are determined by equations (2.108), by satisfying the conditions

$$\sum_{i=1}^{n}[a_{i1}\Delta x_1(0)+\ldots+a_{in}\Delta x_n(0)]^2 \leqslant A; \tag{2.111}$$

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} \neq 0, \tag{2.112}$$

at the initial moment of time $t = t_0$, satisfy the condition

$$\sum_{i=1}^{n}[a_{i1}\Delta x_1(t)+\ldots+a_{in}\Delta x_n(t)]^2 \leqslant A \tag{2.113}$$

in finite time interval $[t_0, t_0 + \tau]$, where A is a sufficiently small positive integer; otherwise unperturbed aircraft motion is *unstable* (i.e., $\tau = 0$). The above definition of motion stability in a finite time interval is credited to G. V. Kamenkov. /154

Stability (or instability) of aircraft motion in a finite time interval can be determined with the aid of the following theorems of G. V. Kamenkov.

Theorem 1. If characteristic equation (2.110), without having multiple roots, has only negative roots or complex ones with negative real parts, unperturbed motion is *stable* in some finite time interval $[t_0, t_0 + \tau]$ independently of functions $\Delta a_{ik}(t)$ and nonlinear terms ΔX_i in equations (2.108). In this case unperturbed motion is stable even during constantly acting perturbations.

Theorem 2. If at least one positive root or two with positive real parts are present among roots of characteristic equation (2.110), unperturbed motion is unstable in a finite time interval (i.e., $\tau = 0$) independently of functions $\Delta a_{ik}(t)$ and nonlinear terms ΔX_i in equations (2.108).

Theorem 3. If there is at least one zero root or two purely imaginary roots, the remainder being negative or complex roots with positive real parts, among the roots of characteristic equation (2.110), unperturbed motion *cannot be stable* in a finite time interval.

Theorem 4. If characteristic equation (2.110), not having positive roots or complex roots with positive real parts, has only multiple real negative roots s_i (where $i = 1, \ldots, \alpha$) and multiple complex roots with negative real parts λ_j (where $j = 1, \ldots, \beta$) and if here all diagonal minors of the determinants

$$D(s_l)=\begin{vmatrix} -s_l & -\frac{1}{2} & 0 & \ldots & 0 & 0 \\ -\frac{1}{2} & -s_1 & -\frac{1}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -\frac{1}{2} & -s_l \end{vmatrix} \quad (l=1,\ldots,\alpha) \tag{2.114}$$

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$$D(\lambda_j)=\begin{vmatrix} -\lambda_j & -\frac{1}{2} & 0 & \ldots & 0 & 0 \\ -\frac{1}{2} & -\lambda_j & -\frac{1}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -\frac{1}{2} & -\lambda_j \end{vmatrix} \quad (j=1,\ldots,\beta) \tag{2.115}$$

are greater than zero, unperturbed motion is *stable* in a finite time interval, independently of functions $\Delta a_{ik}(t)$ and nonlinear terms in equations (2.108).

If one diagonal minor of determinants (2.114) and (2.115) turns out to be negative, unperturbed motion is *unstable* in a finite time interval, independently of functions $\Delta a_{ik}(t)$ and nonlinear terms of equations (2.108).

In certain problems of aircraft dynamics it is better from the practical point of view to set up and solve problems on aircraft *technical stability*.

2.4.7. Calculation of Aircraft Motions by Poincaré's Small Parameter Method

In many practical problems of aircraft motion in corresponding differential equations it is possible to separate certain terms or a group of terms which can be considered small in comparison with the remaining (main) terms. By discarding such small terms and solving the problem for *a simplified system*, it is possible then to use the derived solution as a zero-order approximation for solving the basic problem, by applying a specially developed apparatus for successive approximations. This method is the basis of Poincaré's method.

In flight mechanics *H. Poincaré's small parameter method* is used for calculating and studying periodic and quasi-periodic aircraft motions. The method was first developed by Poincare in connection with investigation of a classical stellar mechanics problem of three bodies. At the present time, in spite of substantial limitations imposed on the choice of equations and on problems which have been set up, the method nevertheless covers many important cases, answers a number of practical questions and is successfully applied in reverse problems of flight mechanics, for example for calculating the orbits of spacecraft.

In order to discover periodic aircraft motions with the aid of Poincaré's method, initial equations of aircraft motion must be represented in the following form:

$$\dot{x}_i = X_i(x_1, \ldots, x_n; t; \mu) \qquad (i = 1, \ldots, n), \tag{2.116}$$

for which the following assumptions are valid:

1) n is the order of the system; /156

2) $x_1, \ldots, x_n$ are the variables which characterize the motion and condition of the aircraft;

3) the X_i functions are holomorphic functions of $x_1, \ldots, x_n, \mu$, i.e., they can be expanded in convergent (at least for small variables) power series in terms of $x_1, \ldots, x_n, \mu$;

4) for a nonautonomous system, i.e., explicit relation between functions X_i and time t, the coefficients of the above-mentioned expansions are also functions of t; in this case the functions must be continuous periodic

functions of time t with some common real period T; the method has also been used for studying autonomous systems when the X_i functions do not explicitly contain time t;

5) the dependence of X_i functions on some sufficiently small parameter μ is a basic assumption so that for some $\mu = \mu_0$ (for example, $\mu_0 = 0$) the initial system (2.116) becomes an equation (or system of equations) with a standard solution $x_i^{(0)}(t)$, for a linear equation (or system of linear equations) with constant coefficients. Later it is assumed that the standard solution is $x_i^{(0)} = \phi_i(t)$ for $\mu_0 = 0$.

With the given assumptions system (2.116) can be presented in form

$$x_i = X_i^{(0)}(x_1, \ldots, x_n; t) + \mu X_i^{(1)}(x_1, \ldots, x_n; t) + \\ + \mu^2 X_i^{(2)}(x_1, \ldots, x_n; t) + \ldots (i = 1, \ldots, n), \tag{2.117}$$

where $X_i^{(0)}$, $X_i^{(1)}$ do not depend on μ.

System (2.117) converts to the system

$$\dot{x}_i^{(0)} = X_i^{(0)}(x_1^{(0)}, \ldots, x_n^{(0)}; t) \quad (i = 1, \ldots, n), \tag{2.118}$$

where $\mu = 0$, which is called the *generating system*, whose solution $x_i^{(0)}$ is called the *generating solution*.

If the generating system (2.118) is a system of linear equations the basic system (2.117) is called quasilinear.

The essence of the method consists in the following theorem.

Poincare's theorem. If generating system (2.118) permits holomorphic solution $x_i^{(0)}(t)$ (i = 1, ..., n), basic system (2.117) has a solution which can be represented as the series

$$x_i = \sum_{k=0}^{\infty} \mu^k x_i^{(k)}(t) \qquad (i = 1, \ldots, n) \tag{2.119}$$

by whole positive powers of μ, which are absolutely and uniformly convergent with sufficiently small modulus μ.

For a selected (fixed) μ, the series (2.119) will converge on a holomorphic function for all values of t which satisfy the condition $t_0 \leqslant t \leqslant \tau$, where $\tau = \tau(\mu)$. In the general case the smaller the modulus μ, the larger is the value of τ. /157

The set of terms

$$x_i^{(0)}(t)+\mu x_i^{(1)}(t)+\ldots+\mu^k x_i^{(k)}(t) \tag{2.120}$$

in (2.119) for fixed k comprises the k-th *approximation* of the solution for x_i.

Discussed below, in accordance with Poincaré's method and the above assumptions, is the problem of the construction of such periodic (or quasi-periodic) solutions of basic system (2.117), which differ slightly from the generating periodic solution $x_i^{(0)} = \phi_i(t)$ of generating system (2.118) for a sufficiently small μ, or more accurately, the problem of such periodic (or quasi-periodic) solutions of system (2.117), which approach the corresponding solutions of system (2.118) for $\mu \to 0$.

Of all possible cases covered by initial system (2.116), the so-called *nonresonance* case is examined, the condition of which is the absence of zero and purely imaginary roots in the characteristic equation of system (2.116) (see Section 2.4.1).

Procedure for Constructing Solution. The construction of periodic solutions for x_i, i.e., of series (2.119), is further examined for the case of non-autonomous systems, i.e., for cases when the right side of equations (2.116) of X_i are periodic, relative to t, functions of period T. It is also assumed that generating system (2.118) has an aperiodic solution

$$x_i^{(0)}=\varphi_i(t) \qquad (i=1, \ldots, n), \tag{2.120'}$$

which corresponds to initial conditions $x_i^{(0)} = \phi_i(t_0)$ and which has period T, so that $\phi_i(t + T) \equiv \phi_i(t)$.

In accordance with Poincaré's theorem basic system (2.117) has solution $x_i = \phi_i(t, \mu)$, represented in the form of the series

$$x_i=\varphi_i(t; \beta_1, \ldots, \beta_n; \mu)=\varphi_i(t)+\sum_{j=1}^{n}\left(\frac{\partial \varphi_i}{\partial \beta_j}\right)_{\beta_j=0}\beta_j+ \left(\frac{\partial \varphi_i}{\partial \mu}\right)_{\mu=0}\mu+\ldots; \qquad (i=1, \ldots, n), \tag{2.121}$$

where β_j denotes deviations of the initial values of x_j and the sought-for solution from the initial values of the same in the generating solution, i.e.,

$$\beta_i = \varphi_i(t_0, \mu) - \varphi_i(t_0) \qquad (i = 1, \ldots, n). \tag{2.122}$$

Series (2.121) will converge absolutely and uniformly for all μ which are sufficiently small in modulus and for all t which satisfy the condition $t \leqslant \tau(\mu)$, if $T \leqslant \tau(\mu)$, then series (2.121) will converge for all values of t. /158

In order that solution $x_i = \phi_i(\beta_1, \ldots, \beta_n; \mu; t)$ be periodic of period T, it is necessary and sufficient that conditions

$$\varphi_i(t_0 + T; \beta_1, \ldots, \beta_n; \mu) - \varphi_i(t_0; \beta_1, \ldots, \beta_n; \mu) = \psi_i = 0 \qquad (i = 1, \ldots, n). \tag{2.123}$$

be satisfied.

Equations (2.123) are used for determining the unknown initial values of the sought-after solution. Consequently, the problem of the existence of periodic solutions for x_i of the basic system (2.117) in form (2.121) reduces to the problem of the solvability of equation system (2.123) in terms of β_i, so that in order for β_i to vanish when $\mu = 0$, since the sought-after solution must convert to a generating one for $\mu = 0$.

On the basis of the theorem of existence of implicit functions equations (2.123) have one and only one solution for $\beta_i = \beta_i(\mu)$ $(i = 1, \ldots, n)$ for sufficiently small μ, while $\beta_i(0) = 0$, and this solution will be analytical relative to μ for the condition

$$\left[\frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(\beta_1, \ldots, \beta_n)}\right]_{\beta_1 = \ldots = \beta_n = \mu = 0} \neq 0. \tag{2.124}$$

Note. In order to calculate the determinant in expression (2.124) for $\beta_1 = \ldots = \beta_n = \mu = 0$, it is sufficient to find only the terms of the expansion of x_i which are linear relative to β_i and μ.

Substitution of solution $\beta_i = \beta_i(\mu)$ in function $\phi_i(t; \beta_1, \ldots, \beta_n; \mu)$ yields one and only one periodic solution for $x_i = \phi_i(t, \mu)$ of basic system (2.117); this solution is analytical relative to μ and converts to a generating one for $\mu = 0$.

Calculation of the sought-after periodic solution in form (2.119), or in equivalent form (2.121) is done in the following manner.

By substituting series (2.119) into equation system (2.117) and equating the coefficients with identical powers of μ in the right-hand and left-hand sides of the equations, it is possible to derive the following equations for successive determinations of functions $x_i^{(k)}(t)$:

for k = 0

$$\dot{x}_i^{(0)} = X_i^{(0)}(x_1^{(0)}, \ldots, x_n^{(0)}; t); \tag{2.125}$$

for k = 1

$$\dot{x}_i^{(1)} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 x_j^{(1)} + \mu X_i^{(1)}(x_1^{(0)}, \ldots, x_n^{(0)}; 0; t), \tag{2.126}$$

where $\frac{\partial X_i}{\partial x_j}$ are the values of $\frac{\partial X_i}{\partial x_j}$ when $x_i = x_i^{(0)}(t)$; /159

for any k

$$\dot{x}_i^{(k)} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 x_j^{(k)} + \Phi_{ik}(x_1^{(0)}, \ldots, x_i^{(k-1)}; t), \tag{2.127}$$

where the functions Φ_{ik} depend only on functions determined from the preceding equation systems.

In this manner the k-th approximation is determined by a system of linear equations of the type

$$\dot{y}_i = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 y_j \qquad (i = 1, \ldots, n) \tag{2.128}$$

with coefficients periodic relative to t.

In problems of airplane flight mechanics cases are often encountered in which the initial system (2.116) permits the existence of a first integral, for example of the type $\Phi(x_1, \ldots, x_n; \mu; t) = C$, which is equivalent to identical equality of zero of determinant (2.124). In such cases equations (2.123) will be dependent and it will be necessary to exclude one of the equations from examination, for example, $\psi_n = 0$. Then solution of the problem of the existence of periodic spacecraft trajectories reduces to proof of the existence of a simple solution of the equation system

$$\psi_1=\psi_2=\ldots=\psi_{n-1}=0 \tag{2.129}$$

relative to $\beta_1, \ldots, \beta_{n-1}$, for which it is sufficient to require that

$$\left[\frac{\partial(\psi_1, \ldots, \psi_{n-1})}{\partial(\beta_1, \ldots, \beta_{n-1})}\right]_{\beta_1=\ldots=\beta_{n-1}=\mu=0} \neq 0. \tag{2.130}$$

One value of β_i (for example, β_n) can be taken arbitrarily.

If initial system (2.116) permits the existence of several first integrals, then condition (2.130) will vary in the corresponding manner.

If the initial equation system for motion (2.116) is autonomous, i.e., has the form

$$\dot{x}_i=X_i(x_1, \ldots, x_n; \mu), \tag{2.131}$$

then determination of periodic aircraft trajectories with an arbitrary period is possible, and the periods of the generating solution and solution of the system (2.131) do not necessarily have to be equal.

In a number of cases the use of Poincaré's method may be greatly facilitated with the aid of the following theorem.

The Theorem of Symmetry. If for differential equation system /160

$$\ddot{x}_i=X_i(x_1, \ldots, x_n; \dot{x}_1, \ldots, \dot{x}_n) \tag{2.132}$$

the right-hand sides are such that the system itself is invariant relative to the substitution

$$\left.\begin{aligned} &x_i \to x_i; \quad \dot{x}_i \to -\dot{x}_i; \quad t \to -t; \quad (i=1, \ldots, s; \quad j=s+1, \ldots, n), \\ &x_j \to -x_j; \quad \dot{x}_j \to \dot{x}_j \end{aligned}\right\} \tag{2.133}$$

then the conditions of periodicity of the solution will acquire the form

$$\dot{x}_l(0)=0;\quad \dot{x}_l\left(\frac{T}{2}\right)=0 \quad (l=1,\ldots,s);$$
$$x_j(0)=0;\quad x_j\left(\frac{T}{2}\right)=0 \quad (j=s+1,\ldots,n), \tag{2.134}$$

where T is the period of solution.

A Small Parameter. Poincaré's method which we have illustrated, imposes definite limitations on the magnitude of the parameter μ. These limitations result, first of all, from Poincaré's theorem which demands sufficient degree of smallness of the parameter for convergence of series (2.119); secondly, from the tendency to be limited in practice by the lower approximations during calculation of the terms of series (2.119) (for example, by the first approximation), and this is related to the sufficient smallness of parameter μ for a given accuracy of calculation.

However, in each specific practical problem, solved by Poincare's method, the parameter μ has a real, physical content, being a function of the design, aerodynamic, thrust and other characteristics. Therefore, we cannot arbitrarily consider the parameter μ as small as we like without changing the physical meaning of the problem under consideration.

When parameter μ, in the physical sense of the problem, may acquire the values $0 \leqslant \mu \leqslant \mu_1$, the following problems arise:

First, does the magnitude of μ_1 satisfy the conditions of convergence of series (2.119)?

Second, is the value for μ_1 such that the first approximation ensures a given accuracy of calculation?

If the answers are no, it follows that one should refrain from using Poincare's method; with a negative response to the second question and a positive response to the first question, one should proceed to calculating subsequent approximations.

2.4.8. Calculation of Aircraft Motions by Krylov-Bogolyubov Averaging Method /161

In many practical problems of flight mechanics aircraft motion equations can be presented as the following system:

$$\ddot{z}_s+\lambda_s^2 z_s=\mu Z_s(z_1,\ldots,z_n;\dot{z}_1,\ldots,\dot{z}_n;t;\mu) \quad (s=1,\ldots,N), \tag{2.135}$$

which, in turn, with the aid of a substitution of variables

$$z_s = x_s e^{j\lambda_s t} + x_{-s} e^{-j\lambda_s t};$$
$$\dot{z}_s = j\lambda_s x_s e^{j\lambda_s t} - j\lambda_s x_{-s} e^{-j\lambda_s t} \qquad (2.136)$$

can be converted to the so-called *standard* form, from the standpoint of the averaging method:

$$\dot{x}_s = \mu X_s(x_1, \ldots, x_n; t; \mu) \quad (s = 1, \ldots, n), \qquad (2.137)$$

where $x_1, \ldots, x_n$ are variables which characterize the motion and condition of the aircraft; μ is a sufficiently small real parameter, and $X_1, \ldots, X_n$ are functions which are holomorphic relative to the variables $x_1, \ldots, x_n, \mu$, i.e., can be written as a series by whole positive powers of the indicated variables, in particular

$$X_s(x_1, \ldots, x_n; t; \mu) = X_s^{(1)}(x_1, \ldots, x_n; t) + \mu X_s^{(2)}(x_1, \ldots, x_n; t) + \ldots \qquad (2.138)$$

In view of expansion (2.138), it is possible to represent system (2.137) in the form

$$\dot{x}_s = \mu X_s^{(1)}(x_1, \ldots, x_n; t) + \mu^2 X_s^{(2)}(x_1, \ldots, x_n; t) + \ldots \quad (s = 1, \ldots, n), \qquad (2.139)$$

or

$$\dot{x} = \mu X(x, t, \mu) = \mu X_1(x, t) + \mu^2 X_2(x, t) + \ldots, \qquad (2.140)$$

where $x = (x_1, \ldots, x_n)$; $X_k = (X_1^{(k)}, \ldots, X_n^{(k)})$ are real vector functions $(k = 1, 2, \ldots)$.

In motion equations (2.140) or initial equations (2.135) the terms which contain parameter μ play the role of perturbing factors, which act on the supporting or *unperturbed aircraft motion*, determined by equations (2.140) for $\mu = 0$. In this sense equations (2.140) or (2.135) when $\mu \neq 0$ can be treated as equations of *perturbed aircraft motion*.

If for functions $X_k(x, t)$ with sufficiently small parameter μ there is an average value for the explicitly entering t: /162

$$\bar{X}_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T X_k(x, t)\, dt, \tag{2.141}$$

then the Krylov-Bogolyubov averaging method can be used for constructing asymptotic solutions for equations (2.140) in the form

$$x = \bar{x}(t) + \mu u_1(\bar{x}, t) + \mu^2 u_2(\bar{x}, t) + \ldots + \mu^k u_k(\bar{x}, t), \tag{2.142}$$

where

$$\dot{\bar{x}} = \mu A_1(\bar{x}) + \mu^2 A_2(\bar{x}) + \ldots + \mu^k A_k(\bar{x}). \tag{2.143}$$

In practice the applicability of these solutions is determined not by the properties of the convergence of sums (2.142) and (2.143) for $k \to \infty$, but by their asymptotic properties for $\mu \to 0$ for any fixed k. The asymptotic properties of the solutions are included in the requirement that expression (2.142) with small μ quite accurately represent the solution of equations (2.140) for a quite prolonged interval of time.

For fixed k equations (2.142) are called *solutions of the k-th approximation*, and equations (2.143) are called *averaged equations of the k-th approximation*. For construction of asymptotic solution of a k-th approximation in the form of expansion (2.142) the averaging method thus determines the functions

$$u_1(\bar{x}, t), \ldots, \quad u_k(\bar{x}, t); \quad A_1(\bar{x}), \ldots, A_k(\bar{x}),$$

so that expression (2.142) satisfies equation (2.140) with an accuracy up to the order of magnitude $\mu^{(k+1)}$.

The analogous method of constructing asymptotic solutions is used for systems of the more general type, whose mathematical model can be presented as

$$\begin{aligned} \dot{x} &= \mu X_1(x, y, t) + \mu^2 X_2(x, y, t) + \ldots; \\ \dot{y} &= Y_0(x, y, t) + \mu Y_1(x, y, t) + \mu^2 Y_2(x, y, t) + \ldots, \end{aligned} \tag{2.144}$$

where all symbols correspond to equation (2.140) or to system (2.137) and, moreover, $x = (x_1, \ldots, x_n)$, $X_k = (X_1^{(k)}, \ldots, X_n^{(k)})$ are n-dimensional vector functions ($k = 1, 2, \ldots$); $y = (y_1, \ldots, y_m)$, $Y_k = (Y_1^{(k)}, \ldots, Y_m^{(k)})$ are m-dimensional

vector functions (k = 0, 1, 2, ...), while x are slowly changing variables, because $\dot{x} \sim \mu$, and y are quickly changing variables, since $\dot{y} \sim 1$; X_k and Y_k, respectively satisfy conditions for t of the type (2.141). /163

Note. In the simplest case limit (2.141) exists if $X_k(x, t)$ in equation (2.140) [or $X_k(x, y, t)$ and $Y_k(x, y, t)$ in system (2.144)] are periodic or nearly periodic functions of t.

Construction of solutions. Results of the averaging method are given later for the case when functions $X_k(x, t)$ are functions which are periodic with respect to t with period 2π, which allow expansion in Fourier series, i.e., are written in the form

$$X_k(x, t) = \sum_n X_{k,n}(x) e^{jnt} \qquad (k = 1, 2, \ldots). \tag{2.145}$$

In this first approximation relative to parameter μ equation (2.140) with consideration of condition (2.145) acquires the form

$$\dot{x} = \mu \bar{X}_1(x) + \mu \sum_{n \neq 0} X_{1,n}(x) e^{jnt}, \tag{2.146}$$

where

$$\bar{X}_1 = \lim_{T \to \infty} \frac{1}{T} \int_0^T X_1(x, t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} X_1(x, t)\, dt = X_{1,0}. \tag{2.147}$$

The process of successive determination of the functions u_1, u_2, ..., A_1, A_2, ... developed in the averaging method for construction of asymptotic functions in the form of expansions (2.142) and (2.143) yields the following results:

$$A_1(\bar{x}) = \bar{X}_1(\bar{x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X_1(\bar{x}, t)\, dt; \tag{2.148}$$

$$u_1(\bar{x}, t) = \sum_{n \neq 0} \frac{X_{1,n}(\bar{x})}{jn} e^{jnt} + u_{1,0}(\bar{x}); \tag{2.149}$$

$$A_2(\bar{x}) = \bar{X}_2(\bar{x}) + \sum_{n \neq 0} \frac{\partial X_{1,-n}(\bar{x})}{\partial \bar{x}} \cdot \frac{X_{1,n}(\bar{x})}{jn} - \frac{\partial u_{1,0}(\bar{x})}{\partial \bar{x}} \bar{X}_1(\bar{x}); \tag{2.150}$$

where $u_{k,0}(\bar{x})$ are arbitrary functions which appear during integration in terms of t at a definite stage. Theoretically the choice of the functions $u_{k,0}(\bar{x})$ is not important, but additional conditions for their unambiguous determination are usually imposed in practical applications, resulting from attempts to effectively solve a specific problem.

In this manner any number of functions u_1, u_2, ...; A_1, A_2, ... can be successively determined and their determination does not pose serious difficulties. However, in view of the rapid complication of formulae (2.147)-(2.150) in practice they are normally limited to some lower approximations, often the first, less often the second. /164

Accuracy of Method. In the general case the averaging method is not concerned with the problem of convergence of expansion (2.142) for $k \to \infty$ and it views expression (2.142) and (2.143) as formal expansions which are necessary for constructing asymptotic approximations.

From the fact that the asymptotic (approximate!) solution of the k-th approximation satisfies equation (2.140) [or system (2.144)] with an error on the order of μ^{k+1}, it has been established using a majorizing procedure that deviation of the approximate solution from the corresponding exact one (when there is congruence of initial conditions) will be limited to a value of the order $\mu^{k+1}t$, which remains small for quite large values of μt (if only the magnitude of μ itself is quite small).

Example 9. This example concerns calculation of the trajectory of the center of mass of a space vehicle by the Krylov-Bogolyubov averaging method when the space vehicle has an initial circular Earth orbit. The motion of the space vehicle is studied by the following assumptions:

1) the space vehicle is viewed as a physical point of mass m;

2) the motion of the space vehicle occurs in the equatorial plane of the Earth;

3) forces of Earth's gravity $G = mg_0 (R_0^2)/R^2$ and constant tangential thrust $F = mf$ act on the space vehicle.

For the given assumptions the equations of motion of the center of mass of the space vehicle in the polar coordinate system, combined with the plane of the trajectory, acquire the form

$$\ddot{R} - R\dot{\varphi}^2 = -g_0 \frac{R_0^2}{R^2} + f \sin \Phi;$$

$$R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = f \cos \Phi,$$

where R is the distance between the space vehicle and the center of attraction at any point of the trajectory; R_0 is the value of R when $t = t_0$; ϕ is the polar angle at any point of the trajectory; g is the acceleration of Earth's gravitational pull for given R(t); $g = g_0$ when $R = R_0$; Φ is the angle between the tangent to the trajectory and the normal to the radius R at any point of the trajectory; f is the permanent tangent to the trajectory of thrust acceleration.

Considering the definitions

$$x = \frac{R_0}{R}; \quad z = \frac{R^2\dot{\varphi}}{R_0^2\dot{\varphi}_0} = \frac{\dot{\varphi}}{x^2\dot{\varphi}_0}; \quad \mu = \frac{f}{g_0}$$

and the relations

$$\dot{R} = -z\dot{\varphi}_0 R_0 \frac{dx}{d\varphi};$$

$$\ddot{R} = -z\dot{\varphi}_0^2 x^2 R_0 \left(z\frac{d^2x}{d\varphi^2} + \frac{dz}{d\varphi}\cdot\frac{dx}{d\varphi} \right);$$

$$V^2 = z^2\dot{\varphi}_0^2 R_0^2 \left[x^2 + \left(\frac{dx}{d\varphi} \right)^2 \right];$$

$$\sin\Phi = \frac{-\frac{dx}{d\varphi}}{\left[x^2 + \left(\frac{dx}{d\varphi} \right)^2 \right]^{1/2}}; \quad \cos\Phi = \frac{x}{\left[x^2 + \left(\frac{dx}{d\varphi} \right)^2 \right]^{1/2}},$$

where

$$V^2 = \dot{R}^2 + R^2\dot{\varphi}^2; \quad \sin\Phi = \frac{\dot{R}}{V}; \quad \cos\Phi = \frac{R\dot{\varphi}}{V},$$

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the equations of motion of the spacecraft are converted to the form

$$z\frac{d^2x}{d\varphi^2} + \frac{dz}{d\varphi}\frac{dx}{d\varphi} + zx = \frac{1}{z} + \mu\frac{\frac{dx}{d\varphi}}{x^2 z\left[x^2 + \left(\frac{dx}{d\varphi} \right)^2 \right]^{1/2}};$$

$$\frac{dz}{d\varphi} = \mu\frac{1}{x^2 z\left[x^2 + \left(\frac{dx}{d\varphi} \right)^2 \right]^{1/2}}$$

or to the equation system

$$\left.\begin{aligned} &\frac{dx}{d\varphi} = y; \quad \frac{dy}{d\varphi} = -x + \frac{1}{z^2}; \\ &\frac{dz}{d\varphi} = \mu\frac{1}{x^2 z\left[x^2 + y^2 \right]^{1/2}}. \end{aligned}\right\} \qquad \text{(I)}$$

The above-illustrated Krylov-Bogolyubov averaging procedure is used for system (I).

The zero approximation relative to μ denotes solution of system (I) for μ = 0, i.e., for z = const:

$$x = \frac{1}{z^2} + r\cos(\varphi + \theta); \quad y = -r\sin(\varphi + \theta),$$

where r and θ are integration constants. Assuming that z = z(ϕ), r = r(ϕ), θ = θ(ϕ), by differentiating x and y in terms of ϕ, substituting the result into system (I), comparing the left and right sides of the obtained equation and solving them relative to dr/dϕ and dθ/dϕ, we find

$$\frac{dr}{d\varphi} = \frac{2}{z^3}\cos(\varphi + \theta)\frac{dz}{d\varphi}; \quad \frac{d\theta}{d\varphi} = -\frac{2}{z^3 r}\sin(\varphi + \theta)\frac{dz}{d\varphi},$$

where

$$\frac{dz}{d\varphi} = \frac{\mu z^5 [1 + rz^2\cos(\varphi + \theta)]^{-2}}{\{[1 + rz^2\cos(\varphi + \theta)]^2 + [rz^2\sin(\varphi + \theta)]^2\}^{1/2}}.$$

By expanding the expression for dz/dϕ in a series in powers of rz^2, assuming that $rz^2 = 0(\mu)$ and retaining the corollary of this in expansion of only those terms which contain rz^2, we obtain equations of motion of the spacecraft for the given problem in standard form according to Krylov-Bogolyubov

$$\frac{dr}{d\varphi} = 2\mu z^2\cos(\varphi + \theta)[1 - 3rz^2\cos(\varphi + \theta)];$$

$$\frac{d\theta}{d\varphi} = -2\mu\frac{z^2}{r}\sin(\varphi + \theta)[1 - 3rz^2\cos(\varphi + \theta)]; \qquad /165$$

$$\frac{dz}{d\varphi} = \mu z^5[1 - 3rz^2\cos(\varphi + \theta)].$$

By applying the averaging operation to the standard equations, we obtain averaged equations of first approximation

$$\left.\begin{aligned}
\frac{dr}{d\varphi} &= \frac{2\mu z^2}{2\pi}\int_0^{2\pi}(1 - 3rz^2\cos\tau)\cos\tau d\tau = -3\mu r z^4;\\
\frac{d\theta}{d\varphi} &= -\frac{2\mu z^2}{2\pi r}\int_0^{2\pi}(1 - 3rz^2\cos\tau)\sin\tau d\tau = 0;\\
\frac{dz}{d\varphi} &= \frac{\mu z^5}{2\pi}\int_0^{2\pi}(1 - 3rz^2\cos\tau)\,d\tau = \mu z^5.
\end{aligned}\right\} \qquad \text{(II)}$$

The second equation of system (II) yields $\theta = \theta_0$. The solution of the latter equation can be written in the form $\frac{1}{z} = (1 - 4\mu\varphi)^{1/4}$.

By combining this solution with the first equation of system (II), we obtain the equation

$$\frac{dz}{dr} = -\frac{z}{3r},$$

which has the solution

$$r_0 r z^3 = 1.$$

The arbitrary constants r_0 and θ_0 are determined from the following initial conditions: when $t = t_0$

$$x = 1; \quad \frac{dx}{d\varphi} = 0; \quad z = 1;$$

$$\frac{dz}{d\varphi} = \mu; \quad \varphi = 0.$$

Consequently,

$$r_0 = -\frac{1}{2\mu}; \quad \theta_0 = \frac{\pi}{2}.$$

By combining the results obtained we get the following equation of spacecraft trajectory in first approximation:

$$x = \frac{R_0}{R} = (1 - 4\mu\varphi)^{\frac{1}{2}} + 2\mu(1 - 4\mu\varphi)^{\frac{3}{4}} \sin\varphi.$$

In this equation the first term $(1 = 4\mu\phi)^{1/2}$ describes the so-called "main motion" of the spacecraft, which is spiral in nature. The second term complements the "main motion," determining the oscillations of the center of mass of the spacecraft about the separate "main motion" in a spiral trajectory.

CHAPTER 3. METHODS OF OPTIMIZATION OF CONTROL AND DESIGN PARAMETERS OF AIRCRAFT

BASIC DEFINITIONS /167

α_ρ - control parameter.
a - vector of the control parameters.
H, H - Hamilton function (Hamiltonian problem of optimization).
L - Lagrange function.
J - Quality criterion.
m - number of control variables u_j (j = 1, 2, ..., m).
n - number of variables of state (phase coordinates) x_i (i = 1, 2, ..., n).
r - number of control parameters α_ρ (ρ = 1, 2, ..., r).
t - independent variable (argument), time.
R^n - n-dimensional space with real components.
U^m - m-dimensional tolerable range of change of control variables.
u_j - control coordinate.
u - control vector (control).
X^n - n-dimensional tolerable range of change of variables of state x.
x_i - variable of state (phase coordinate).
x - vector of state (phase vector).
λ_i - conjugate variable (Lagrange factor).
λ - vector of conjugate variables (vector of Lagrange factors).

Indices

"0" - beginning of process (initial conditions).
"1" - end of process (terminal conditions).
"T" - index of vector or matrix transposition.

Symbols

$\arg\max_{u \in U^m} H(u)$ - value of vector u which maximizes function H on set U^m.
$\arg\min_{u \in U^m} H(u)$ - value of vector u which minimizes function H on set U^m.
$\max_{u \in U^m} H(u)$ - maximum of function H for variables u, which are elements of set U^m.

$\min_{u \in U^m} H(u)$ - minimum of function H for variables u, which are elements of set U^m.

δ - symbol of variation.

$\in$ - symbol of membership of an element in some set (for example, $u \in U^m$ means that u is an element of set U^m or belongs to set U^m). /168

$\subset$ - symbol of inclusion (for example, $X^n \subset R^n$ means that the set X^n is contained in set R^n or set X^n is included in set R^n).

In the general process of aircraft design two types of problems are encountered:

1. Planning the system of flight operation, aimed at attaining the stated goal (formation of trajectories, flight conditions and profiles, and selection of control methods achieves these trajectories; etc.). This group of problems can be called the planning of the aircraft motions.

2. Planning of aerodynamic, structural and strength systems (choice of geometric, aerodynamic, structural and other parameters), assuring the attainment of general practical flight characteristics and specific flight operations. This class of planning problems involves the choice of resources required for attaining the stated goals.

Aircraft motion planning is closely related to the second group of problems, since information obtained during motion planning is the initial (and largely determining) information for these problems. But in cases when an operating aircraft already exists (i.e., available resources are determined), optimizing changes can be made during its modification.

Both the traditional optimal problems of flight mechanics (calculation of maximum flight velocities, best ascent conditions, maximum range, flight time, optimal turn, etc.), and the new problems arising in connection with the development of aerospace technology (insertion into orbit, transfer from orbit to orbit, reentry into the atmosphere, orientation and stabilization in space, etc.) are most effectively and rigorously solved at this time on the basis of general methods of mathematical theory of optimal control processes.

The importance of the mathematical theory of optimal control processes consists in the fact that it provides a standard methodology for solving an extremely large class of optimal planning and control problems, eliminates empiricism and insufficient generality of former partial methods and facilitates the exchange of valuable results and methods acquired in allied fields.

Optimal processes theory gives the solution for a large class of practical problems of flight mechanics in sufficiently general statement with consideration of the majority of technical limitations imposed on the feasibility of

the flight. The importance of methods of optimal processes theory has increased particularly in recent years in connection with the extensive adoption of multi- /169
purpose high-speed computers in the planning process, and also in connection with the utilization of on-board and ground-based high-speed control systems.

The fundamental concepts of mathematical theory of optimal processes, the mathematical statement of various types of optimization problems and general solution methods, based on the application of the maximum principle, dynamic programming and classical variational calculus, are discussed in this chapter. Also given are examples of the application of these principles to certain flight mechanics and flight control problems.

It should be pointed out that problems of optimization of the motion of controlled aircraft, from the standpoint of mathematical description, can be classified as deterministic, stochastic and game (conflict) problems. In this chapter emphasis is placed on deterministic problems.

3.1. General Description of Optimization Methods Based on Mathematical Theory of Optimal Control Processes

3.1.1. Technical Problem of Optimal Control and Mathematical Model

The initial information for the solution of optimal control problems is contained in the statement of a technical problem. A technical flight control problem can be formulated in interesting (nonformal), often nebulous terms. The use of mathematical methods requires concise and rigorous formulation of a problem to eliminate possible uncertainties and ambiguities and simultaneously render the problem mathematically correct. For this purpose a technical problem requires adequate mathematical formulation, called the mathematical model of a technical optimization problem.

Mathematical model. A mathematical model is a sufficiently complete mathematical description of a physical system and control process within the framework of the selected extent of approximation and detailization.

A mathematical model transforms the initial technical problem into some mathematical system, in the final analysis, into some number system. In this system, are explicitly stated (enumerated) on the one hand, all information, without which it is impossible to embark on the analytical or numerical analysis of the problem, and on the other hand, any additional information derived from the technical essence of the problem and which reflects specific requirements on its characteristics.

A complete mathematical model of a technical control optimization problem consists of several partial mathematical models, such as mathematical model of /170
the controlled motion of an aircraft and forces acting upon it, mathematical model of available resources and technical limitations, mathematical model of the quality criterion of the control process, mathematical model of control actions, etc.

Thus the mathematical model of a technical control problem is characterized by a set of defined mathematical relations among its elements (differential equations, constraints of the equality and inequality type, quality functions, initial and boundary conditions, etc.). General conditions are established in optimal processes theory, which the elements of the mathematical model must satisfy in order that the corresponding mathematical optimization problem be, first, clearly defined and, second, meaningful, i.e., not contain conditions leading to the absence of a solution. These conditions are presented in part in sections 3.2 and 3.3.

It should be pointed out that the technical formulation of a problem and its mathematical model do not remain invariant during the process of analysis, but exist in relationship with each other (Figure 3.1). The initial technical formulation and its mathematical model usually experience substantial changes at the end of the analysis. Thus the formulation of an adequate mathematical model resembles the iteration process, during which both the statement of the technical problem itself and the formulation of the mathematical model are refined. It should be stressed that the mathematical model of a given technical problem may not be unique (for instance, the problem can be formulated in different coordinate systems, etc.). Therefore the variation of the mathematical model for which the solution and analysis of the problem are simplest, must be found.

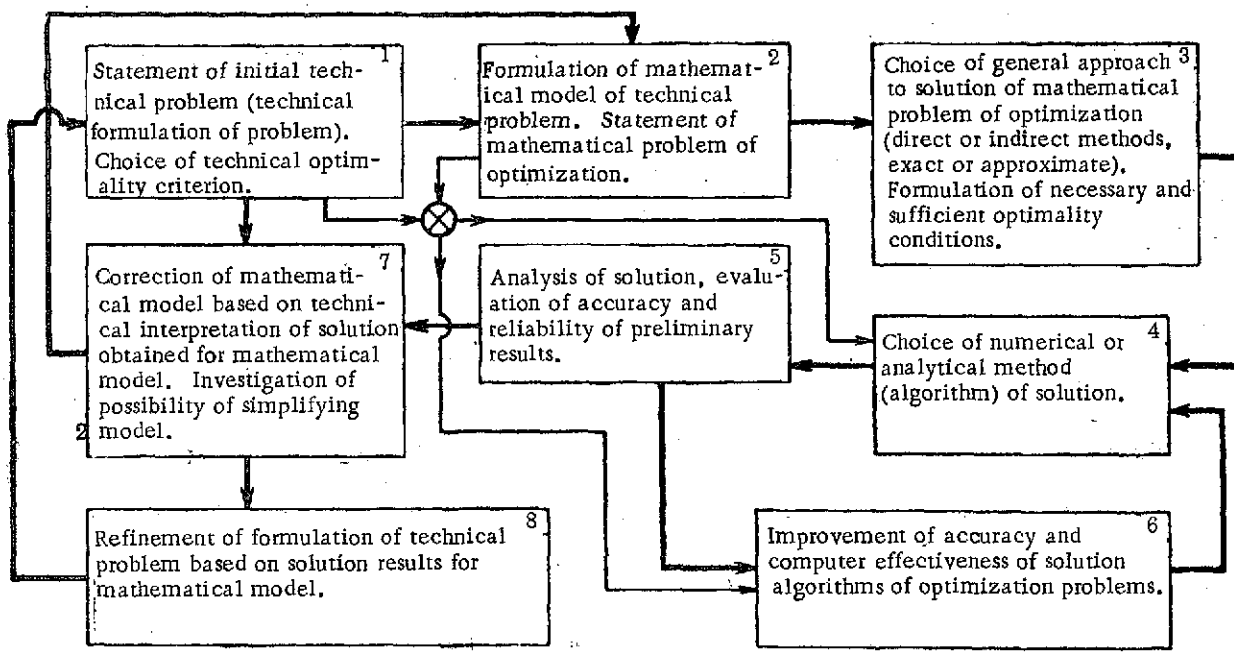
1
Statement of initial technical problem (technical formulation of problem). Choice of technical optimality criterion.
2
Formulation of mathematical model of technical problem. Statement of mathematical problem of optimization.
3
Choice of general approach to solution of mathematical problem of optimization (direct or indirect methods, exact or approximate). Formulation of necessary and sufficient optimality conditions.
4
Choice of numerical or analytical method (algorithm) of solution.
5
Analysis of solution, evaluation of accuracy and reliability of preliminary results.
6
Improvement of accuracy and computer effectiveness of solution algorithms of optimization problems.
7
Correction of mathematical model based on technical interpretation of solution obtained for mathematical model. Investigation of possibility of simplifying model.
8
Refinement of formulation of technical problem based on solution results for mathematical model.

Figure 3.1. Diagram of the Interconnection of the Formulation of a Technical Optimization Problem with the Corresponding Mathematical Model and Results of the Solution of the Optimization Problem for a Mathematical Model.

The choice of optimality criterion is an important step in the statement and solution of a technical control problem. This choice is an informal act, need not be prescribed by some theory, and is determined entirely by the content of the problem. In some cases the formal expression of the technical concept of systems optimality permits several equivalent (nearly equivalent) formulations. In such cases the success and simplicity of the solution depends largely on the selected optimality criterion (assuming that it adequately reflects the requirements of the technical problem on the system in all cases). After a mathematical model of a control process has been constructed it is further analyzed and optimized by mathematical methods.

3.1.2. Classification of Methods of Optimal Processes Theory /172

The methods of optimal processes theory can be classified conditionally as indirect and direct methods.

Indirect Methods. Indirect methods reduce a problem of optimization of dynamic characteristics of a system, which are functionals (see section 3.2.5) to the solution of known mathematical problems. The indirect methods include the following:

1) L. S. Pontryagin's maximum principle (see section 3.4.3) and the Lagrange factors method of classical variational calculus (see section 3.9.2). The maximum principle reduces the solution of the problem of functional optimization to the solution of theoretically known problems of maximization (or minimization) of some special function of a finite number of variables in combination with the solution of the boundary problem for a canonical system of ordinary first-order differential equations. In classical variational calculus the problem of functional optimization amounts to the solution of the boundary problem for a system of ordinary second-order differential equations. The maximum principle is particularly useful for the solution of flight mechanics problems because of features of notation of differential equations of motion of aircraft[5] and the possibility of most simply considering various types of limitation on the magnitude of control and phase variables. Classical variational calculus is more helpful in problems described by more general differential equations than in flight mechanics, in particular, in problems that cannot be solved in terms of derivatives, and which do not contain limitations as inequalities on the control and phase variables;

2) R. Bellman's dynamic programming optimality principle (see section 3.5.2) and the Hamilton-Jacobi method of classical variational calculus. In these methods the problem of functional optimization amounts to the solution

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[5] Formulation of the maximum principle pertains to problems described by systems of ordinary first-order differential equations that can be solved in terms of the derivatives. It is specifically in this form that the equations of motion of aircraft are most often written.

of one nonlinear differential equation in first-order partial derivatives with one boundary condition;

3) certain methods based on the use of results of functional analysis.

Direct methods. The direct methods of optimal processes theory reduce the problem of functional optimization to the construction of a minimizing (or maximizing) series, on whose basis, using passage to the limit, the exact solution to the problem can be found. The direct methods include those based on the reduction of functional optimization problems to problems on the conditional extremum of functions of a finite number of variables, different variations of gradient methods (see section 4.3.1), Ritz-Galerkin methods, etc. /173

In either case, when indirect or direct methods are used, the final solution of an optimization problem can be sought either in analytical (closed) form or in numerical form.

Analytical solutions. Analytical solutions (with the exception of rare cases such as, for instance, linear stationary systems with quadratic quality criterion) can be found only for problems in simplified statements. With them it is possible to analyze the qualitative features of optimal control of an investigated aircraft. If the analytical solution is not too unwieldy the required technical conclusions can be derived from it. Since solutions of this type do not depend on the specific numerical values of the parameters of a system or on boundary conditions, they are highly universal. However, in problems whose statements approach an actual technical situation, it is either impossible, as a rule, to find the solution in closed form, or it leads to extremely complex expressions. In this case numerical solution methods should be employed.

Numerical methods. In the present stage of development of computer mathematics the numerical methods exhibit generality comparable to that of the analytical methods. The results pertaining to numerical methods of solving optimization problems are presented in Chapter 4.

3.1.3. Necessary Control Optimality Conditions, Sufficient Control Optimality Conditions and the Problem of Existence of Optimal Control

The necessary control optimality conditions set forth in the following section were derived for various types of optimization problems on the basis of analytical indirect methods (see section 3.1.2.) of optimization and comprise a set of functional relations, which the extremal solution must satisfy. They were derived in the assumption of existence of optimal control (optimal solution), which assumption is important for subsequent application. In other words, if an optimal solution exists, then it must satisfy the reduced (and therefore necessary) conditions. However, other solutions, not optimal, may satisfy the very same necessary conditions [like the necessary condition $\partial f(x)/\partial x = 0$ for the maximum function of one variable is satisfied by the points of maximum and points of inflection of the function $f(x)$]. Therefore,

if the solution satisfies the necessary optimality conditions it still is not necessarily optimal. /174

The use of necessary conditions alone makes it possible in theory to find all solutions which satisfy them, and then to choose from among them those which actually are optimal. In practice, however, it is usually not possible to find all solutions which satisfy the necessary conditions in view of the enormous amount of work this process involves. Therefore when any solution is found that satisfies the necessary conditions it is advisable to check whether or not it is actually optimal in terms of the initial statement of the problem.

The analytical conditions which, when satisfied on the solution, guarantee its optimality, are called sufficient conditions of control optimality. The formulation of these conditions and especially their practical (for example computer) checking are often extremely difficult tasks. Some of the sufficient conditions of optimality are presented in section 3.4.4.

In general the use of necessary optimality conditions would be more justified if the fact of existence or existence and uniqueness of optimal control could be established for the examined problem. This question is mathematically very complex.

The problem of existence of optimal control consists of two parts:

1) existence of permissible control (i.e., control belonging to a given class of functions), which satisfies the given limitation and converts the system from the given initial state to given terminal state (see section 3.2.10). Sometimes the boundary conditions of a problem are chosen such that the system, in view of limitation of its energy resources, is in no condition to satisfy them, i.e., the existence of even one permissible control is not certain. In this case the optimization problem has no solution;

2) existence of optimal control among permissible controls and its uniqueness.

The second aspect, like the first, of this problem in the case of nonlinear systems of general form has not been solved with completeness sufficient for applications. The problem is exacerbated by the fact that uniqueness of optimal control does not mean uniqueness of control which satisfies the necessary conditions. The fact is that some most important necessary condition (most often the maximum principle) is usually satisfied.

Checking of further necessary conditions is a rather awkward task. This indicates the importance of any information about the uniqueness of controls to satisfy the necessary optimality conditions, by information about specific properties of such controls. /175

Conclusions of the existence of optimal control on the basis of the fact that the physical problem is solved must be treated very carefully. Indeed it is necessary to work with a mathematical model using the methods of optimal

processes theory. The existence of a solution for the mathematical model is also the necessary condition of adequate description of the physical process by the mathematical model. Since various simplifications are used in the formulation of the mathematical model, the influence of which on the existence of solutions is difficult to predict, proof of existence is an independent mathematical problem.

Thus:

1) from the existence of optimal control derives the conclusion that at least one control exists which satisfies the necessary optimality conditions. The existence of a control which satisfies the necessary optimality conditions does not mean that optimal control exists;

2) from the existence of optimal control and uniqueness of control, satisfying the necessary conditions derives the conclusion of uniqueness of optimal control. Uniqueness of optimal control does not mean that the equation that satisfies the necessary optimality conditions is unique.

3.1.4. General Presentation of Results Obtained by Methods of Optimal Control Theory

Optimal processes theory is the foundation of the only method for planning optimal motions of aircraft. By using the methods of optimal processes theory on problems of flight mechanics the following results can be obtained:

1. Time program, optimal in terms of some criterion, of change of control actions and optimal constants of control (planning, adjustments) parameters with consideration of various limitations on their magnitudes.

2. Optimal trajectories, flight conditions and profiles with consideration of limitations on their range in space.

3. Optimal control laws as feedback, determining the structure of the control system (solution of the problem of control synthesis).

4. Limiting flight characteristics or other quality criteria which are then used as standards for comparison with other systems.

5. Solutions of boundary problems of passing from one point in phase space into another; in particular the problem of guidance to an uncontrolled vehicle. /176

6. Optimal guidance strategies for pursuit of controlled vehicle.

3.1.5. Conditions of Efficient Application of Optimization Methods

Control optimization methods are effectively applied:

1. In complex systems, where it is difficult to find suitable solutions on the basis of experience. Experience indicates that optimization of small

systems can lead to large errors in the quality criterion of a combined system. It is better to find an approximate solution of the problem of optimization of a system as a whole (for example in simplified statement) than the exact solution for an independent subsystem.

2. In new problems, where there is lack of experience in the formulation of satisfactory control process characteristics. In such cases the formulation of the optimal problem often makes it possible to establish the qualitative character of control.

3. In the earliest possible planning stage, when there is great freedom of choice. After determination of a large number of design solutions a system becomes inflexible and subsequent optimization may not be productive.

4. When it is necessary to determine the directions of change of control and parameters that produce the greatest change in the quality criterion (determination of quality gradient).

It should be pointed out that when applied to well-known and time tested systems optimization methods may yield little gain, since the practical solutions based on experience are usually close to the optimal. Thus, in traditional problems of flight mechanics (optimal ascent, maximum range), in the case of free boundary conditions for most of the variables optimal controls usually yield a gain of 5 to 12% compared with previous controls. In the case of fixed boundary conditions the gain may reach 20 to 50%.

In certain practical problems of flight mechanics there is a certain "coarseness" of optimal controls and parameters, i.e., small changes of quality criteria correspond to large local changes of controls and parameters. This sometimes gives rise to the opinion that optimums are always smooth in practice and rigorous optimization methods are not necessary.

The fact is that "coarseness" of control occurs only when optimal control corresponds to a stationary point of the quality criterion. In this case a change in control by ε results in deviation of the quality criterion by a magnitude of the order ε^2.

The stated coarseness may also not occur in the case of controls on the boundary of the tolerable range. This property should be investigated for each specific problem. Furthermore, in some problems even slight improvements of the quality criterion, gained through optimization, may be very important (in space flight mechanics, for example, a 0.5% increase in payload or final velocity may cut the cost of a system considerably). /177

Complex control optimization problems often impose extreme requirements on the characteristics of the computers used for this solution. Therefore it is advisable to use effective simplifying assumptions in the statement of the problem in order that the volume of computations not exceed the capacity of modern computers and not require excessive computer time.

3.2. Fundamental Concepts and Definitions of Mathematical Theory of Optimal Control Processes

3.2.1. Mathematical Models

Optimal process control theory deals with mathematical models of technical problems of optimization of control of physical systems. The mathematical model is a sufficiently complete summarization of the functional relations describing the basic properties of physical systems, functioning and control processes within the framework of the given degree of approximation and detailization, reflecting all important requirements on specific technical properties of the system.

The mathematical model of the technical problem of optimization of aircraft control consists of several partial mathematical models, including the mathematical model of the controlled process (controls, for example, of motion of an aircraft and its control drives), mathematical model of technical limitations on the magnitudes of control actions and on the possible location of the aircraft and its trajectory, mathematical description of the effectiveness (quality) criterion of the control process, etc.

The main elements of a general mathematical model of the technical problem of control process optimization are described in Table 3.1.

The mathematical problem of control process optimization is considered completely defined (correctly stated) if all elements of the mathematical model figuring in Table 3.1 are accurately described. The mathematical model of the aircraft control optimization problem should be constructed following the sequence of the general scheme indicated in this table.

TABLE 3.1. STAGES OF CONSTRUCTION AND ELEMENTS OF MATHEMATICAL MODEL OF TECHNICAL PROBLEM OF CONTROL OPTIMIZATION PROCESS FOR DETERMINISTIC SYSTEMS WITH LUMPED PARAMETERS AND CONTINUOUS TIME

No. of Stage	Description of Stage	Elements of Mathematical Model (Result of Analysis or Choice)	Comments
I	Informal description of technical problem and analysis; choice and substantiation of degree of accuracy and detailization of description of the system by physical theories. Physical statement of problem.	Formulation of theoretical case or of narrow problem of analysis in general terms. Determination of physical laws to which various aspects of the problem are subordinated.	Data on which are based the construction of the mathematical model and formulation of specific assumptions permitting the use of mathematical simplifications, are prepared.

TABLE 3.1. (CONTINUED)

No. of Stage	Description of Stage	Elements of Mathematical Model (Result of Analysis or Choice)	Comments
II	Formulation of mathematical model. Mathematical statement of problem.		Accomplished on basis of Stage I.
	Choice and listing of variables of state (phase coordinates), their domains and time intervals in which it is desirable to examine the controlled process. Choice of coordinate system (or systems) in which it is desirable to examine motion and control processes.	Vector of state (phase vector) $x = (x_1, x_2, \ldots, x_n)^T$ $x \in \tilde{X}^n \subset R^n$ (n is the dimensionality of phase space R^n). Domain of vector x: $\tilde{X}^n$ Time interval $T = \{t, t_0 \leq t \leq t_1\}$	Choice of phase coordinates for specific problem not unique (depends, for example, on choice of coordinate system).
	Determination of general laws to which evolution of state of examined system is subordinated. Evaluation of range of application (domain).	Differential equations of motion $\frac{dx}{dt} = f(t, x, y)$ $f = (f_1, f_2, \ldots, f_n)^T$ Domain of f: $t \in T$, $x \in \tilde{X}^n$, $y \in Y^{m_1}$	Here y is the vector of still undefined elements in right side of equations of motion.
	Choice and listing of control variables and their domain, and also of control parameters and perturbations.	Control variables $u = (u_1, u_2, \ldots u_m)^T$, $u \in U^m \subset R^m$ Control (design) parameters $a = (a_1, a_2, \ldots, a_r)^T$, $a \in \tilde{A}^r \subset R^r$ Perturbations $w = (w_1, w_2, \ldots, w_s)^T$, $w \in \tilde{W}^s \subset R^s$, $m + r + s = m_1$	Vector of undefined elements y either becomes control u or known function (t, x), or control parameter a. In stochastic problems w are random functions.

TABLE 3.1. (CONTINUED)

No. of Stages	Description of Stage	Elements of Mathematical Model (Result of Analysis or Choice)	Comments
	Analysis of technical limitations on magnitudes of control actions, phase coordinates and control parameters.	Limitations of equality type $\phi(t, x) = (\phi_1, \phi_2, \ldots, \phi_\mu)^T = 0$ $\chi(t, x, u, a) = (\chi_1, \chi_2, \ldots, \chi_\nu)^T = 0.$ Limitations of inequality type $\psi(t, x) = (\psi_1, \psi_2, \ldots, \psi_{\mu_1})^T \geq 0$ $\kappa(t, x, u, a) = (\kappa_1, \kappa_2, \ldots, \kappa_{\nu_1})^T \geq 0$	Sometimes limitations are given in the form $u \in U^m \subset \tilde{U}^m$ $x \in X^n \subset \tilde{X}^n$ $a \in A^r \subset \tilde{A}^r$, where U^m, X^n, A^r are closed bounded ranges.
	Choice of functional classes for controls and trajectories. Determination of tolerable trajectories, controls and control parameters.		Usually u(t) are piecewise continuous, bounded functions of time t, x(t) are continuous piece-wise smooth functions of time t.
	Formulation of initial and boundary conditions (target of evolution of system).	Condition of type $g(t_0, t_1, x(t_0), x(t_1), a) = (g_1, g_2, \ldots, g_l)^T = 0$ $(l \leq 2n + 2 + r)$ or $h(t_0, x(t_0), a) = (h_1, h_2, \ldots, h_{l_1})^T = 0$ $q(t_1, x(t_1), a) = (q_1, q_2, \ldots, q_{l_2})^T = 0$	Free boundary conditions are also formulated.

TABLE 3.1. (CONTINUED)

No. of Stages	Description of Stage	Elements of mathematical Model (Result of Analysis or Choice)	Comments
	Choice of control quality criterion, aimed at reaching stated goal.	Various functionals $J[u, a]$, defined on solutions of system $\frac{dx}{dt} = f(t, x, u, a)$	
	Choice of computer operator (max, min, max min, min max,...), application of which for quality criterion is mathematical expression of technical concept of systems optimality. Determination of arguments of this operator (u, a, t, etc.), formulation of optimization problem.	Example, $\max_{u \in U} J[u]$ $\min_{u \in U} J[u]$ $\min_{u \in U} \max_{t \in T} J[u]$ $\min_{u \in U} \max_{w \in W} J[u, w]$	
III	Correction of technical statement of problem.	Number of variables, form of equations, criteria, boundary conditions, etc.	Analytical difficulties of analyzing model as formulated may compel further simplification.
	Equivalent transforms of mathematical model for convenience of analytical and numerical methods of solving optimization problems.	Conversion to new phases and (or) control variables, boundary conditions, etc.	In particular, use of methods of penalty functions, reduction to similar problems, etc.
	Change of mathematical model for convenience of computations. Formulation of concepts "practically optimal system" and "Practical accuracy of results" in specific problems.		Accomplished on the basis of general (Stage I) and mathematical (Stage II) formulations of problem

3.2.2. Variables of State (Phase Coordinates) of Controlled Process /182

The mathematical model of a controlled process is built upon the foundation of the mathematical model of the technical problem of control process optimization. This model, in turn, is based on the concept of variables of state (phase coordinates), which are introduced in the problem in the following manner.

Suppose a physical controlled system S (aircraft, its control drives, etc.) can be idealized to the extent that at each given moment of observation time $t = t'$ on observation interval $T = \{t,\ t_0 \leq t \leq t_1\}$, $t' \in T$, its properties can be described by a finite set of real numbers $x_1(t'), x_2(t'), \ldots, x_n(t')$, which are regarded as components of some vector $\bar{x}(t') = (x_1(t'), x_2(t'), \ldots, x_n(t'))^T$.

As the time of observation changes, generally speaking, vector x also changes. This change can be caused by actions applied to the object. If and only if $t > t'$ (t' is an arbitrary moment on observation interval T) the properties of the system, as before, are completely described by the vector

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \qquad (3.1)$$

and if the number n is the least number of values of $x_i(t')$, with the aid of which it is possible to predict changes of x(t) for all $t > t'$ on the basis of known values $x(t')$ and applied actions, known on T, then the vector x(t) is defined as the vector of state of (deterministic) system S at moment t (or vector of phase coordinates).

The values of x_i are called components of state or phase coordinates.

Since x_i change with time, sometimes $x_i(t)$ are called variables of state (phase variables).

The set of all possible states $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ at different moments of time $t \in T$ comprise n-dimensional space of states $X^n \subset R^n$ (n-dimensional phase space). The point $x \in X^n$ is the mapping point of this state. Vector $z = (x, t)^T$, i.e., the state at moment t, is called an event (phase).

The set of all possible events z comprises space $Z^{n+1} \subset R^{n+1}$ of events. Point $z \in Z^{n+1}$ is the mapping point of the space of events.

Comments

1. In flight mechanics the variables of state are usually the spatial and angular coordinates of the aircraft, linear and angular velocities, mass, etc. The choice of variables of state is often determined by the choice of coordinate system in which aircraft motion is examined. Hence it is clear that different

phase coordinates can be used for describing the same problem, i.e., their choice is not unique. However all conceivable sets of phase coordinates are equivalent in terms of description of the state of a system.

2. The phase space of optimal control theory never coincides with the phase space of configurations and pulses of mechanics, since it may contain coordinates of nonmechanical systems (electrical and hydraulic, for example). /183

3.2.3. Control

System S is called controlled on interval $[t_0, t_1]$ if its behavior $t > t_0$ depends only on the initial state ($t = t_0$, $x_0 = x(t_0)$), being the behavior of some variable vector u (of the input of the system):

$$\mathbf{u} = (u_1, u_2, \ldots, u_m)^{\tau}, \; m \geqslant 1, \tag{3.2}$$

called the control vector (or simply control) and of constant vector a:

$$\mathbf{a} = (a_1, a_2, \ldots, a_r)^{\tau}, \; r \geqslant 0, \tag{3.3}$$

called the vector of control (design) parameters.

Vector u acquires values from some set U^m of m-dimensional space R^m with coordinates $u_1, u_2, \ldots, u_m$. This set may be the entire space R^m or part of it $U^m \subset R^m$. In flight mechanics U^m is usually a closed region of space R^m. /184

Set U^m is called the set of tolerable controls. Some forms of sets U^m are illustrated in Figure 3.2.

Constant vector a of control parameters in flight mechanics usually belongs to some closed set $A^r \subset R^r$ (R^r is a space with coordinates $a_1, a_2, \ldots, a_r$).

Comments

In flight mechanics the control variables u_j are usually the coordinates of deflection of control surfaces, thrust vector, etc., and control parameters a_ρ are the design, weight and geometric characteristics of the aircraft (its design parameters). The closure and boundedness of sets U^m and A^r denote that in actual constructions the positions of the control surfaces and design parameters cannot be arbitrarily large. The control variables need not coincide in each specific problem with the coordinates of deflection of the physical "controls" and control elements, but are determined by the degree of detailization of the problem and by the perfection of the mathematical model of the object. In certain "truncated" problems, i.e., problems without sufficiently complete description of the actual object, the coordinates of the angular position of the aircraft, velocity components and directions, etc. can be the control variables.

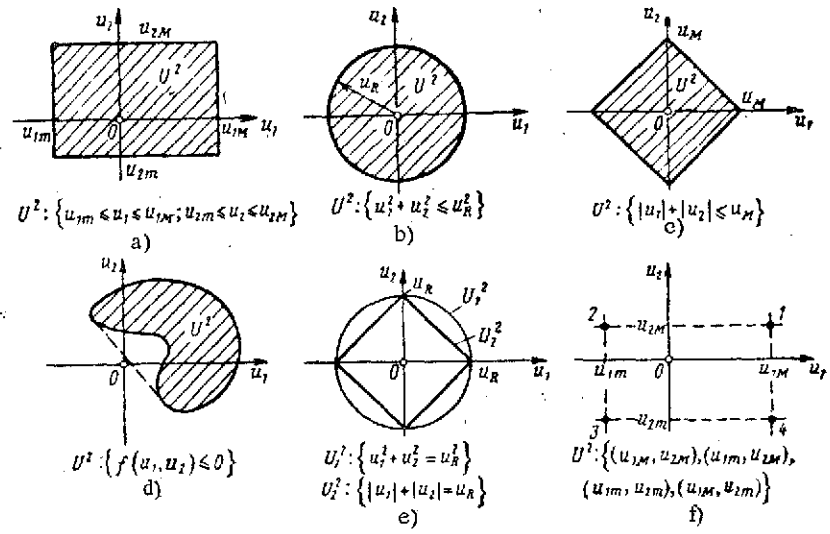

Figure 3.2. Types of Sets U^2 of Tolerable Controls. a, b, c, Closed bounded convex regions containing coordinate origin; d, Open region not containing coordinate origin; e, Non-convex one-dimensional regions U_1^2 and U_2^2; f, Discrete set of tolerable values (1, 2, 3, 4 -- isolated points)

3.2.4. Evolution of State of System. Differential Equations of Motion

A change of state (evolution) of system S on a time interval $T = \{t,\ t_0 \leqslant t \leqslant t_1\}$ normally describes a system of ordinary differential first-order equations with a good degree of approximation in the normal Cauchy form:

$$\frac{dx}{dt} = f(t, x, u, a), \tag{3.4}$$

where $x = (x_1, x_2, \ldots, x_n)^T$ is the vector of state;

$u = (u_1, u_2, \ldots, u_m)^T$ is the control vector;

$a = (a_1, a_2, \ldots, a_r)^T$ is the vector of design parameters;

$f = (f_1, f_2, \ldots, f_n)^T$ is the vector of the generalized force.

System (3.4) forms an essential part of the mathematical model of physical system S (see Table 3.1). In the mathematical model described by the system of differential equations, the presence of derivative dx/dt in the left side of the equation system (3.4) is a formal feature of the variable of state x. The

control variables u enter only in the right side of system (3.4) and are not encountered under the sign of the derivative (this is a formal feature of the control variable).

It is assumed that the vector-function f(t, x, u, a) is defined for all $x \in X^n$, $u \in U^m$, $a \in A^r$, $t \in T$, is continuous for the set of variables t, x, u, a and continuously differentiated for t, x, a. Because the behavior of vector u may be arbitrary (with the exception of condition $u \in U^m$) and, in addition, we may arbitrarily select the constant vector $a \in A^r$, equation (3.4) determines the controlled process. The course of the controlled process will be determined on some interval $t_0 \leqslant t \leqslant t_1$, if vector u is given in one of two forms on this interval: /185

$$u = u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T; \tag{3.5}$$

$$u = v(x, t) = (v_1(x, t), v_2(x, t), \ldots, v_m(x, t))^T. \tag{3.6}$$

The vector-function u(t) is called programmed (temporal) control (or simply control), and the vector-function v(x, t) is called coordinate control, or law of control (in certain problems of flight mechanics, the law of guidance).

The control law (3.6) physically expresses the known feedback principle, according to which the magnitude of the controlling action is determined on the basis of measurement of the current state of system x and, perhaps, moment of time t.

According to (3.4) the time sequence of conditions $x(t, x_0, t_0)$, which is called the phase trajectory (behavior, evolution, motion) of system S corresponds to each choice of vectors for the control parameters a and control u [in form (3.5) or (3.6)]. The pair of vector-functions {u(t), x(t)} or {v(x, t), x(t)} is called the control process.

3.2.5. The Functional. Control Quality Criterion

The value J[u(t)] is called the functional of function u(t) on segment $t_0 \leqslant t \leqslant t_1$ if a certain number is placed in correspondence with each u(t), $t \in [t_0, t_1]$, which belongs to some class of functions.

In this manner, the functional J[u(t)] is a function in which the role of the independent variable (functional argument) is played by the function u(t). In this case J[u(t)] depends on the set of all the values acquired by function u(t) on segment $[t_0, t_1]$, and can be viewed as a function of an infinite number of independent variables.

For each fixed finite moment of time $t_1 = t_1'$ condition $x(t_1')$ of system S, which moves from initial condition (t_0, x_0) in conformity with equation (3.4), is simultaneously a vector functional (i.e., a vector, whose components are functionals) of control u(t) and a vector-function of vector a and a vector of initial conditions $x_0(t_0)$. The quality criteria of the control process are functionals.

A quite general form of the quality criterion in the theory of optimal process has the form

$$J[\mathbf{u}(t); \mathbf{a}] = \Phi(t_0, t_1, \mathbf{x}_0, \mathbf{x}_1; \mathbf{a}) + \int_{t_0}^{t_1} f_0(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{a})\,dt, \tag{3.7}$$

where x(t) satisfies system (3.4); u(t) is some selected control; a is a control parameter. /186

In particular, each one of coordinates $x_i(t_1)$ of system (3.4) can be written in the form (3.7):

$$x_i(t_1) = \int_{t_0}^{t_1} f_i(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{a})\,dt + x_i(t_0) \qquad (i = 1, 2, \ldots, n).$$

3.2.6. Autonomous Systems

If the right-hand sides of system (3.4) and functions f_0 and Φ in (3.7) do not depend explicitly on time the corresponding problem is called autonomous:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{a}); \tag{3.4'}$$

$$J[\mathbf{u}(t), \mathbf{a}] = \Phi(\mathbf{x}_0, \mathbf{x}_1, \mathbf{a}) + \int_{t_0}^{t_1} f_0(\mathbf{x}, \mathbf{u}, \mathbf{a})\,dt. \tag{3.7'}$$

Autonomous systems are invariant relative to displacement on axis t. Therefore only the length of process $t_1 - t_0$ is important for autonomous systems and it may be assumed that $t_0 = 0$.

3.2.7. Permissible Programmed Control

The vector-function u(t) is called permissible programmed control in the basic problem (see Section 3.3.1), if:

a) u(t) belongs to the class of functions which are piece-wise continuous for t, on interval $[t_0, t_1]$, i.e., it may have only a finite number of points of first-order discontinuity;

b) the values u(t) belong to a given set U^m for all $t \in [t_0, t_1]$.

Piece-wise continuous controls correspond to the hypothesis of "zero lag" deflection of control surfaces, which at the moment of discontinuity instantaneously "jump" from one position to the other.

If it is desired to consider the "lag" of the rudders then one should look for control in the class of piece-wise smooth continuous functions u(t). Such a class of permissible controls is sometimes reduced to the preceding by means of inserting a new zero lag control $\bar{u}(t)$ which is connected to the "old" control u(t) by the relation

$$\frac{du}{dt} = \bar{u}; \quad \bar{u} \in \bar{U}^m, \tag{3.8}$$

where

$$u = (u_1, u_2, \ldots, u_m)^T;$$
$$\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)^T.$$

When $\bar{U}^m$ is a closed and bounded region this means that limitations on the values of the first derivatives of the vector-function u(t) are introduced. /188

Because of (3.8) the piece-wise smooth functions u(t) correspond to piece-wise continuous functions $\bar{u}(t)$. In this manner u(t) becomes a variable of state in the new problem and is controlled by means of $\bar{u}(t)$ through system (3.8).

If condition $u \in U^m$ can be discarded from the new problem, the problem reduces to the preceding one for piece-wise continuous control $\bar{u} \in \bar{U}^m$. Otherwise one should turn to the problem of optimization with limitations on the phase coordinates (see Section 3.7).

Figure 3.3 gives examples of equations which belong to the class of piece-wise continuous functions and to other classes.

Comments

Examination of the permissible equations in the class of piece-wise continuous functions is explained by the fact that for optimization of functionals in this class of functions the corresponding mathematical apparatus, the maximum principle, has been developed (see Section 3.4.3).

For each permissible control u(t), by virtue of the assumptions made (see Section 3.2.4) relative to f(t, x, u), there exists a unique absolutely-continuous solution of a system $x(t) = x(t, x_0, t_0)$, which satisfies system (3.4) almost everywhere for $[t_0, t_1]$ [i.e., with the exception of a finite

number of points of discontinuity of function u(t)], and for t - t_0 acquires the given value $x_0 = x(t_0)$.

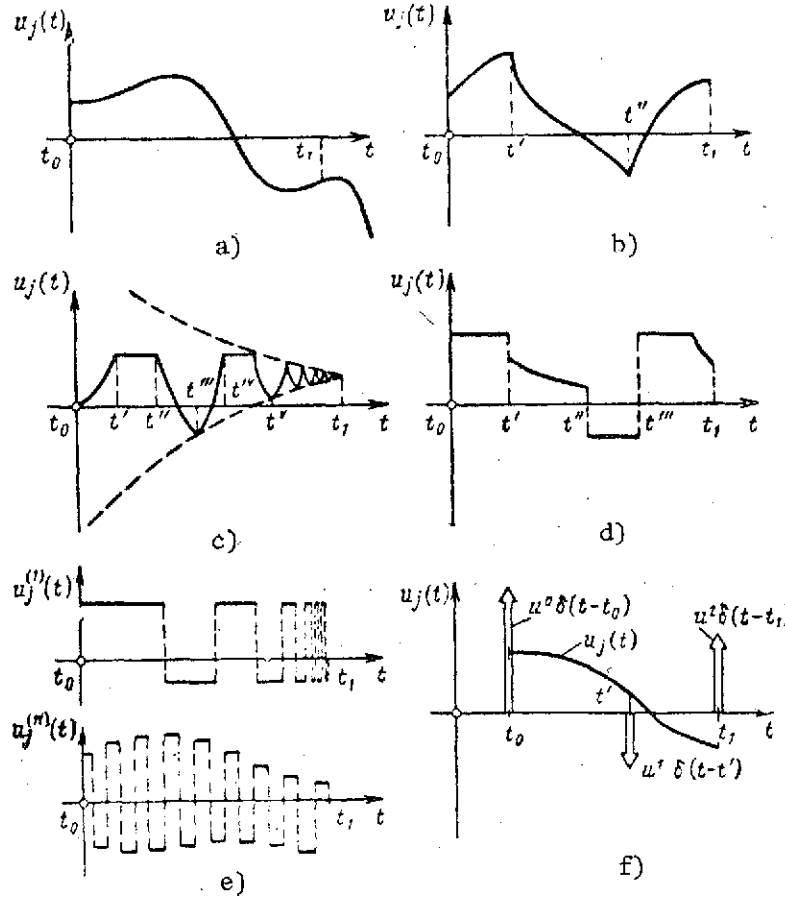

Figure 3.3. Examples of Controls $u_j(t)$, Belonging to Different Classes of Functions.
a, Smooth control; b, Piece-wise smooth continuous control; c, Continuous control (in the vicinity of point t_1 function $u_j(t)$ is undifferentiable); d, Piece-wise continuous control; e, Controls which are not piece-wise continuous ($u_j^{(1)}(t)$, contain an infinite number of change-overs in the vicinity of point t_1; $u_j^{(n)}(t)$ is element of series converging on the function which is broken at each point of the segment $[t_0, t_1]$); f, The control which contains the Dirac δ-function; u^0, u^1, u^2-constants.

3.2.8. Permissible Law of Control

The law of control $v(x, t)$ is permissible for $x \in \tilde{X}^n$, $t \in [t_0, t_1]$, if

a) $v(x, t) \in U^m$ for all $t \in T = [t_0, t_1]$, $x \in \tilde{X}^n$;

b) $v(x(t), t) = u(t)$,

where $x(t)$ is the trajectory of system S;
$u(t)$ is the permissible programmed control for law of control $v(x, t)$.

Vector a of the control parameters is called permissible if its values belong to the given set $A^r \subset R^r$.

3.2.9. Permissible Trajectories and Processes

The phase trajectory $x(t)$ of system S is called permissible if:

a) it is obtained from the solution of a system of differential equations with permissible control $u(t)$ or with permissible law of control $v(x, t)$;

b) the values of $x(t)$ belong to the given region $\tilde{X}^n$ of the space of state X^n.

The control process (x, u) is called permissible if the permissible trajectory $x(t)$ is realized in it due to the action of permissible control $u(t)$ or permissible law of control $v(x, t)$. /189

3.2.10. Boundary Conditions. Boundary Problem

The goal of controlled process (x, u) consists in changing system S from one given initial state $x_0 = x(t_0)$ for $t = t_0$ to given terminal state $x_1 = x(t_1)$ within the period of time $T = t_1 - t_0$. In this case not all the components of vectors x_0 and x_1 and moments of time t_0, t_1 need be fixed; some of them may remain unknown (free). In the general case system S at the initial and terminal moments of time may operate under conditions which are described by equations of the form /190

$$\mathbf{h}(t_0, \mathbf{x}_0, \mathbf{a}) = (h_1, h_2, \ldots, h_{l_1})^T = 0; \tag{3.9}$$

$$\mathbf{q}(t_1, \mathbf{x}_1, \mathbf{a}) = (q_1, q_2, \ldots, q_{l_2})^T = 0 \tag{3.10}$$

or by more general equations of the form

$$\mathbf{g}(t_0, t_1, \mathbf{x}_0, \mathbf{x}_1, \mathbf{a}) = (g_1, g_2, \ldots, g_l)^T = 0, \tag{3.11}$$

where $l_1 + l_2 \leqslant 2n + 2 + r$;
$l \leqslant 2n + 2 + r$.

Equations (3.9)-(3.10) describe (with fixed control parameter a) surfaces of dimensionality $(n + 1 = l_1)$ and $(n - l_2)$ in space (t, x) and are called seperate boundary conditions for the ends of the phase trajectory. Examples of boundary conditions are given in Figure 3.4.

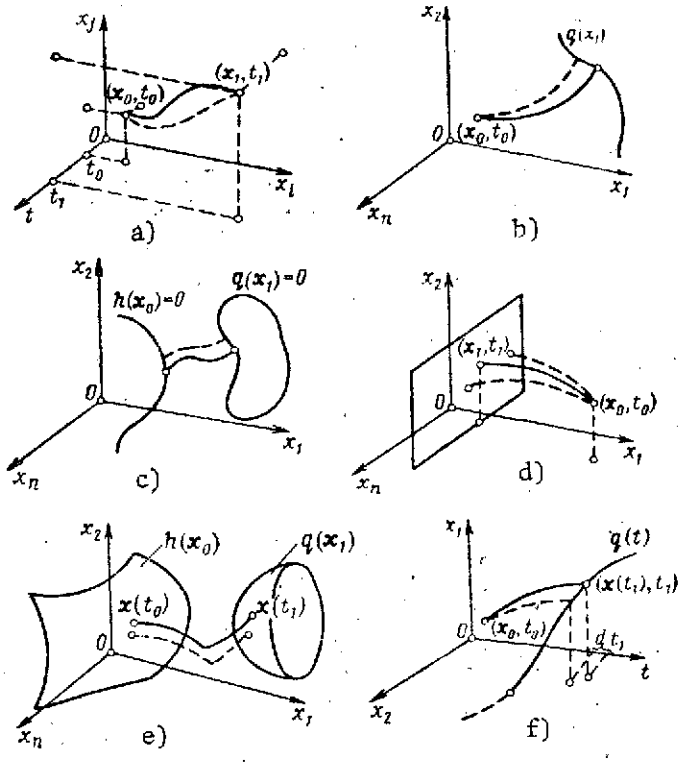

Figure 3.4. Examples of Boundary Conditions. a, Left and right ends of the phase trajectory are attached; b, Left end is attached, right end is free; c, Left and right ends are free; d, Left end is attached, right end is free, with the exception of coordinate x_1; e, General case of free boundary conditions; f, Boundary conditions in a problem of rendezvous (interception). —— Optimal trajectory; ---- Arbitrary trajectory.

Equations (3.11) are called mixed boundary conditions. When the values of the phase coordinates at moment t_0 (or t_1) are not fixed the boundary conditions for the left (or the right) end of the trajectory are called free. Seperate conditions of form (3.9)-(3.10) are often called mobile boundary conditions.

Determination of controls u(t), for which solution of system (3.4) satisfies conditions of the type (3.9)-(3.10), is called a two-point boundary problem.

The change of initial condition x_0 to terminal condition x_1 on given segment $[t_0, t_1]$ is not always possible. However, if only one pair of vectors $\{u(t), a\}$ or $\{v(x, t), a\}$, is found which realizes the indicated change, other pairs of vectors usually also exist which realize this same change. In this case some quality criterion $J[u, a]$ corresponds to each pair $\{u(t), a\}$. It is possible to state the problem of finding such $\{u(t), a\}$, which minimize or maximize this criterion.

3.3. Statement of Basic Problems of Optimal Control

3.3.1. Basic Problem of Optimal Programmed Control

The basic problem of optimal programmed control in the form of time program (3.5) for system (3.4) with quality criterion (3.7) and boundary conditions (3.11) is formulated as follows.

Among all permissible programmed controls $u = u(t) \in U^m$ and control parameters $a \in A^r$ on segment $[t_0, t_1]$, which translate point (t_0, x_0) to point (t_1, x_1), find those for which functional (3.7) in solutions of system (3.4) acquires the smallest (greatest) possible value when condition (3.11) is satisfied. /191

Control u(t), which solves this problem, is called the optimal (programmed) control, and vector a is the vector of optimal parameters.

If the pair $\{u^*(t), a^*\}$ imparts the absolute minimum to functional $J[u(t), a]$ in solutions of system (3.4) the relation

$$J_{\min} = J^* = J[u^*(t), a^*] \leqslant J[u(t), a] \tag{3.12}$$

is satisfied for all $u \in U^m$, $a \in A^r$, which are permissible and which realize the given translation when condition (3.11) is satisfied.

The analogous definition is valid for the absolute maximum (with replacement of the sign of inequality $\leqslant$ by the sign $\geqslant$).

Comments

By definition of the absolute minimum (3.12) it follows that the absolute minimal value of the functional $J^* = J[u^*, a^*]$ is unique, which generally cannot be said for optimal control $u^*(t)$ and optimal parameter a^*.

3.3.2. Basic Problem of Optimal Coordinate Control

The basic problem of optimal coordinate control is known in the theory of optimal processes as the problem of synthesis of the optimal law of control,

and in some problems of flight mechanics it is known as the problem of optimal law of guidance.

The problem of synthesis of the optimal law of control for system (3.4) with quality criterion (3.7) and boundary conditions (3.9), (3.10), where for simplification it is assumed that functions f_0, f, h, q, Φ do not depend on vector a, is formulated as follows.

Among all the permissible laws of control v(x, t) find one such that for any initial conditions (t_0, x_0) from (3.9) during formulation of this law in (3.4) and in (3.7) the given translation is realized in (3.10) and quality criterion J[u] acquires the least (greatest) value.

3.3.3. Optimal Trajectories

The trajectory of system (3.4), which corresponds to optimal control u*(t) or optimal law v*(x, t), is called the optimal trajectory. The set of optimal trajectory x*(t) and optimal control u*(t) forms the optimal controlled process {x*(t), u*(t)}.

It has been established that in the presence of vector a of control parameters in f_0, f, h, q, Φ the problems of programmed and coordinate control are equivalent.

Because the law of optimal control v*(x, t) has the form of the law of control with feedback it remains optimal for all initial conditions (x_0, t) and all coordinates x. In contrast to the law v*(x, t), programmed optimal control u*(t) is optimal only for those initial conditions for which it was calculated. /192
The function u*(t) will change with a change of initial conditions. In this connection the contrast of the law of optimal control v*(x, t) to programmed optimal control u*(t) is important from the point of view of the practical feasibility of a control system, inasmuch as the choice of initial conditions in practice can never be made absolutely exactly.

Properties of Optimal Controls and Optimal Trajectories

1. Any segment of an optimal trajectory (optimal control) is in itself an optimal trajectory (optimal control). This property is formulated mathematically as follows.

Let $u^*(t)$, $t_0 \leq t \leq t_1$ be the optimal control for some functional J[u], which corresponds to change from state (t_0, x_0) to state (t_1, x_1) on optimal trajectory x*(t). The numbers t_0, t_1 and the vector x_0 are fixed, while the vector x_1, generally speaking, is free.

The points $x^*(\tau_0)$ and $x^*(\tau_1)$, which correspond to time $t = \tau_0$ and $t = \tau_1$, where $t_0 \leq \tau_0 \leq \tau_1 \leq t_1$ are selected on the optimal trajectory $x^*(t)$.

Then control $u^*(t)$ on segment $[\tau_0, \tau_1]$ is an optimum which corresponds to the conversion from state $x^*(\tau_0)$ to state $x^*(\tau_1)$, and segment $[x^*(\tau_0), x^*(\tau_1)]$ is an optimal trajectory (Figure 3.5).

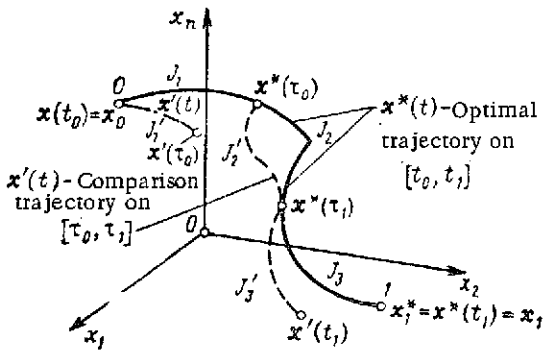

Figure 3.5. Basic Property of Optimal Trajectories.
$J_2' > J_2$; J_i, J_i' ($i = 1, 2, 3$) - Values of Functional on Segments of Optimal Trajectory and on Comparison Trajectories, Respectively.

Thus, if the initial condition of the system is $x^*(\tau_0)$ and the initial moment of time $t = \tau_0$, its optimal subsequent motion will be a segment of trajectory $x^*(t)$, $\tau_0 \leq t \leq t_1$, which is the part of the optimal trajectory between points (t_0, x_0) and (t_1, x_1) regardless of the manner in which the system arrived in this state. This condition is an essential feature of the optimization process and serves as the basis of dynamic programming (see Section 3.5.2).

Comment:

The above brief formulation of the basic property of optimal trajectories should not be too broadly interpreted. The requirement that the initial and terminal points of the trajectories of comparison lie on the optimal trajectory at the same moment of time τ_0, τ_1 as the points of the optimal trajectory, or that the free right end of x_1' of the comparison trajectory end at the same moment t_1 as the end of the optimal trajectory, is essential. If they are not fulfilled this property, generally speaking, does not occur. Thus, if only the initial point $x_0 = x(t_0)$ and the moments of time t_0 and τ_0 are given, and $x(\tau_0)$ is free, the segment of trajectory $x^*(t)$, $t_0 \leq t \leq \tau_0$ cannot be optimal. In this case another segment $x'(t)$ (see Figure 3.5), generally speaking, may be optimal.

2. Autonomous systems (see Section 3.2.6) are invariant relative to displacement on axis t. This means that if $u^*(t)$, $t_0 \leq t \leq t_1$ completes the translation $x_0 \to x_1$ and imparts the value J^* to functional $J[u]$, control $u^*(t + \tau)$, $t_0 - \tau \leq t \leq t_1 - \tau$ also completes the translation $x_0 \to x_1$ and imparts the value J^* to the functional $J[u]$ for all real τ. /193

3.3.4. Geometric Interpretation of Basic Problem of Optimal Control

The basic problems of optimal control with fixed ends may be given the following equivalent geometric formulation.

Let for $t = t_0$ be given the initial condition $x_0 = x(t_0)$ and for $t = t_1$ be given the end condition $x_1 = x(t_1)$, where t_0, t_1, x_0, x_1 are fixed values. Then the term $\Phi(t_0, t_1, x_0, x_1)$ in functional $J[u]$ (3.7) is a known number Φ_0.

We interpolate the new variable x_0, whose law of change has the form

$$\frac{dx_0}{dt} = f_0(t, x, u, a), \tag{3.13}$$

with the initial condition

$$x_0(t_0) = x_{00} = \Phi_0.$$

We connect this variable to system (3.4). Then when $t = t_0$ this system is located at point $(x_0(t_0), x_1(t_0), \ldots, x_n(t_0))^T$, and when $t = t_1$ it is located at point $(x_0(t_1), x_1(t_1), \ldots, x_n(t_1))^T$.

where

$$x_0(t_1) = \Phi_0 + \int_{t_0}^{t_1} f_0(t, x, u, a)\, dt = J[u].$$

And so, if straight line Π (Figure 3.6) is drawn through point $(0, x_1)$ in $(n + 1)$-dimensional space of points (x_0, x) parallel to axis 0_{x0} the solution of system (3.4), (3.13) passes through the point on straight line Π with coordinate $x_0(t_1) = J$ when $t = t_1$. /194

Now the basic problem of optimal programmed control is formulated geometrically as follows.

In $(n + 1)$-dimensional phase space $(x_0, x_1, \ldots, x_n)^T$ are given the following:

1) for $t = t_0$ point (Φ_0, x_0);

2) straight line Π, parallel to axis $0x_0$ and passing through point $(0, x_1)$.

Of all permissible programmed controls $u = u(t)$, with the property that the corresponding solution $(x_0(t), x(t))$ of system (3.4), (3.13) with initial condition $(\Phi_0, x_1(t_0), \ldots, x_n(t_0))^T$ intersects straight line Π when $t = t_1$, find the one for which the point of intersection with straight line Π has the least (greatest) coordinate $x_0(t_1) = J$.

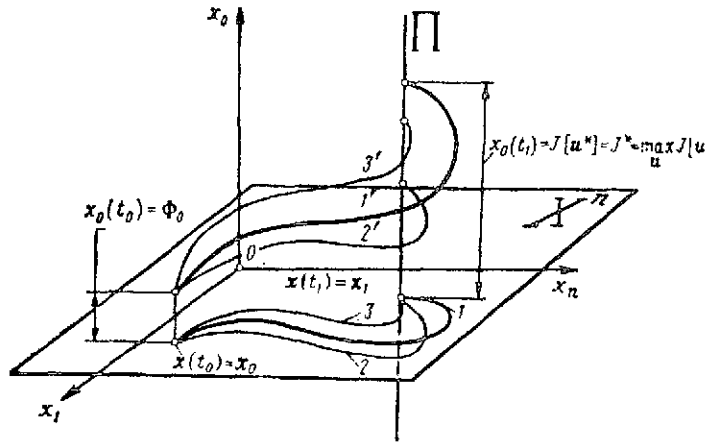

Figure 3.6. Geometric Formulation of Basic Problem of Optimal Control.
1, Optimal trajectory; 1', Change in quality criterion J on optimal trajectory; 2, 3, Nonoptimal trajectories which pass through points (x_0, t_0), (x_1, t_1); 2', 3', Change in quality criterion J on nonoptimal trajectories.

3.4. Necessary Conditions of Optimization for Basic Problem of Programmed Control. The Maximum Principle.

3.4.1. Brief Formulation of the Problem

Let the following be given: 1. System of differential equations of motion

$$\frac{dx}{dt}=f(t, x, u, a), \tag{3.14}$$

where $f(t, x, u, a)$ are determined for all $x = (x_1, x_2, \ldots, x_n)^T \in \tilde{X}^n \subset R^n$, /195
$t_0 \leqslant t \leqslant t_1$, $u \in U^m$, $a \in A^r$, are continuous for set of variables (t, x, u, a) and are continuously differentiable for (t, x, a).

2. Relations which initial (t_0, x_0) and the end (t_1, x_1) phases of motion in system (3.14) satisfy:

$$g_j(t_0, t_1, x_0, x_1, a)=0 \qquad (j=1, 2, \ldots, l<2n+2+r), \tag{3.15}$$

where the functions g_j are continuously differentiable for all of their arguments.

3. Control quality criterion (functional)

$$J[u(t), a] = \Phi(t_0, t_1, x_0, x_1, a) + \int_{t_0}^{t_1} f_0(t, x, u, a)\, dt, \tag{3.16}$$

where Φ and f_0 have all of the necessary derivatives.

The set U^m is a closed and bounded Euclidean region of m-dimensional space R^m. The function u(t) is considered permissible if it is piece-wise continuous and its values belong to set U^m: $u(t) \in U^m$.

3.4.2. Certain Auxiliary Constructions and Terminology

The following are introduced: 1. Time-dependent vector of conjugate coordinates (vector-function of Lagrange factors)

$$\lambda(t) = (\lambda_0(t), \lambda_1(t), \ldots, \lambda_n(t))^T. \tag{3.17}$$

2. Constant vector μ:

$$\mu = (\mu_1, \mu_2, \ldots, \mu_l)^T. \tag{3.18}$$

3. Auxiliary functions (the Hamiltonian of the optimization problem and the Lagrange function)

$$H(t, x, u, \lambda, a) = \sum_{i=1}^{n} \lambda_i f_i(t, x, u, a) + \lambda_0 f_0(t, x, u, a) \tag{3.19}$$

and

$$L(t_0, t_1, x_0, x_1, a, \mu) = \sum_{j=1}^{l} \mu_j \varphi_j(t_0, t_1, x_0, x_1, a) + {}_0\lambda\Phi(t_0, t_1, x_0, x_1, a). \tag{3.20}$$

4. System of differential equations which is conjugate[6] to (3.14), (3.16) and which determines the change of vector $\lambda(t)$: /196

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{k=0}^{n} \lambda_k \frac{\partial f_k(t, x, u, a)}{\partial x_i} \quad (i = 0, 1, 2, \ldots, n). \tag{3.21}$$

---

[6] The system of linear differential equations $\dot{y} = B(t)y$ is called conjugate for the system $\dot{x} = A(t)x + f(t)$, if $B(t) = -A^T(t)$ and the dimensionalities of vectors x and y (and also of matrices B(t) and A(t)) are identical. Thus system (3.21) is in fact conjugate to linearized system (3.14), (3.23):

$$\dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}(t), \hat{u}(t)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\hat{x}(t), \hat{u}(t)} \delta u(t),$$

where $\hat{x}(t)$, $\hat{u}(t)$ are some reference trajectory and a reference control, respectively.

Using function H the original system of equations (3.4) is written in the form

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial \lambda_i}=f_i(t, \mathbf{x}, \mathbf{u}, \mathbf{a}) \qquad (i=0, 1, 2, \ldots, n). \tag{3.22}$$

The new variable $x_0(t)$ corresponds to the index i = 0, which is determined by the scalar equation

$$\frac{\partial x_0}{dt}=f_0(t, \mathbf{x}, \mathbf{u}, \mathbf{a}) \tag{3.23}$$

with the initial condition

$$x_0(t_0)=x_{00}=\Phi(t_0, t_1, \mathbf{x}_0, \mathbf{x}_1, \mathbf{a}). \tag{3.24}$$

The system of equations

$$\left.\begin{aligned}\dot{\tilde{x}}&=\left(\frac{\partial H}{\partial \lambda}\right)^{\mathrm{T}}=\tilde{\mathbf{f}};\\ \dot{\lambda}&=-\left(\frac{\partial H}{\partial \tilde{x}}\right)^{\mathrm{T}}=-\left(\frac{\partial \tilde{f}}{\partial \tilde{x}}\right)^{\mathrm{T}}\lambda,\end{aligned}\right\} \tag{3.25}$$

where $H=\lambda^{\mathrm{T}}\tilde{f}$; $\partial\tilde{f}/\partial x$ is the Jacobi matrix, $\tilde{x}=(x_0, x_1, \ldots, x_n)$, $\tilde{f}=(f_0, f_1, \ldots, f_n)$, $x \in \tilde{X}^{n+1}$ is called the canonical system of differential equations related to the basic problem.

3.4.3. L. S. Pontryagin's Maximum Principle

Let $u^*(t)=(u_1^*(t), \ldots, u_m^*(t))^{\mathrm{T}}$, $t \in [t_0, t_1]$ be a permissible control and $a^*=(a_1^*, a_2^*, \ldots, a_r^*)^{\mathrm{T}}$ be a permissible value of the vector of the parameters, such that the trajectory $x^*(t)$ of system (3.14) which corresponds to them satisfies conditions (3.15) for the ends.

For optimization (in terms of a minimum[7]) of quality criterion (3.16) of control $u^*(t)$, of trajectory $x^*(t)$ and of vector of the control parameters a^* it is necessary to have such a non-zero variable vector $\lambda(t)=(\lambda_0(t), \lambda_1(t), \ldots, \lambda(t))^{\mathrm{T}}$, $\lambda_0(t)=\text{const} \geq 0$[8] and such a constant vector $\mu=(\mu_1, \mu_2, \ldots, \mu_l)^{\mathrm{T}}$ such that the following conditions are satisfied.

---

[7] The case of the maximum of functional J[u, a] is reduced to the problem in the given statement by means of examination of functional $J_1[u, a] = -J[u,a]$.

[8] Usually it may be assumed that $\lambda_0 = 1$, see Section 3.4.4, corollary 2.

1. Vector-functions $x^*(t)$, $u^*(t)$, $\lambda(t)$ and vector a^* satisfy system /197

$$\left.\begin{aligned}\frac{dx_i^*}{dt} &= \frac{\partial H(t, x^*(t), u^*(t), \lambda(t), a^*)}{\partial \lambda_i};\\ \frac{d\lambda_i}{dt} &= -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t), a^*)}{\partial x_i},\\ (i&=0, 1, 2, \ldots, n).\end{aligned}\right\} \tag{3.26}$$

2. The function $H(t, x^*(t), u, \lambda(t), a^*)$ of variable $u \in U^m$ for each value of $t \in [t_0, t_1]$, i.e., for fixed values of x^* and λ and for fixed vector a^* attains the minimum when $u = u^*(t)$:[9]

$$\begin{aligned}H(t, x^*(t), u^*(t), \lambda(t), a^*) &= H^*(t, x^*(t), \lambda(t), a^*) =\\ &= \min_{u \in U^m} H(t, x^*(t), u, \lambda(t), a^*).\end{aligned} \tag{3.27}$$

Thus optimal control is defined as

$$u^*(t) = u^*(t, x^*(t), \lambda(t) a^*) = \arg\min_{u \in U^m} H(t, x^*(t), u, \lambda(t), a^*). \tag{3.28}$$

The maximum principle, consequently, confirms that optimal control $u^*(t)$ at each moment of time t minimizes the projection of phase velocity $\tilde{\dot{x}} = \tilde{f}(t, x, u)$ of the control process (i.e., the projection of velocity of mapping point $\tilde{x} \in \tilde{X}^{n+1}$) in the direction given by vector $\lambda(t)$ (we recall that

$$H = \sum_{i=0}^{n} \lambda_i f_i = \lambda^T \dot{x} = \lambda^T \tilde{f}(t, x, u, a)$$

is the scalar product of vectors $\lambda(t)$ and $\dot{x}$.

3. The conjugate variables $\lambda_i(t)$ and function $H(t, x^*(t), u^*(t), \lambda(t), a^*)$ are continuous on the optimal trajectory (analog of the Erdman-Weierstrass condition of classical calculus of variations, see Section 3.9.2).

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[9] In contrast to the classical formulation of L. S. Pontryagin's maximum principle in the given case the operation max in (3.27) is replaced by min. In conformity with such replacement, necessary condition (3.27) could be called the maximum principle. It should be noted that in the given case $\lambda_0 \geq 0$, while in classical formulation $\lambda_0 \leq 0$.

4. Conditions of transversality. For terminal points (t_0, x_0), (t_1, x_1) and vector of parameters a* with arbitrary variations of terminal points and parameters, the generalized conditions of transversality are satisfied:

$$\left[H\delta t-\sum_{i=0}^{n}\lambda_i\delta x_i\right]\Bigg|_{t_0}^{t_1}+dL+\sum_{\varrho=1}^{r}\int_{t_0}^{t_1}\frac{\partial H}{\partial a_\varrho}\delta a_\varrho dt=0. \tag{3.29}$$

Here dL is the complete variation of function $L(t_0, t_1, x_0, x_1, \mu, a)$, determined by equation (3.20): /198

$$dL=\frac{\partial L}{\partial t_0}\delta t_0+\frac{\partial L}{\partial t_1}\delta t_1+\sum_{i=0}^{n}\frac{\partial L}{\partial x_i(t_0)}\delta x_i(t_0)+$$
$$+\sum_{i=0}^{n}\frac{\partial L}{\partial x_i(t_1)}\delta x_i(t_1)+\sum_{\varrho=1}^{r}\frac{\partial L}{\partial a_\varrho}\delta a_\varrho, \tag{3.30}$$

where δt_0, δt_1, $\delta x_i(t_0)$, $\delta x_i(t_1)$, δa_ρ are derivatives of the variation of end points and parameters.

Generalized conditions of transversality (3.29) with consideration of expression (3.30) lead to the following 2n + 2 + r relations due to the independence of variation δt_0, δt_1, $\delta x_i(t_0)$, $\delta x_i(t_1)$, δa_ρ:

$$\left(-H+\frac{\partial L}{\partial t_0}\right)\Big|_{t_0}\delta t_0=0; \tag{3.31}$$

$$\left(H+\frac{\partial L}{\partial t_1}\right)\Big|_{t_1}\delta t_1=0; \tag{3.32}$$

$$\left(\lambda_i+\frac{\partial L}{\partial x_i}\right)\Big|_{t_0}\delta x_i(t_0)=0 \quad (i=1, 2, \ldots, n); \tag{3.33}$$

$$\left(-\lambda_i+\frac{\partial L}{\partial x_i}\right)\Big|_{t_1}\delta x_i(t_1)=0 \quad (i=1, 2, \ldots, n); \tag{3.34}$$

$$\left(\frac{\partial L}{\partial a_\varrho}+\int_{t_0}^{t_1}\frac{\partial H}{\partial a_\varrho}dt\right)\delta a_\varrho=0 \quad (\varrho=1, 2, \ldots, r). \tag{3.35}$$

If any end condition $x_i(t_0)$, $x_i(t_1)$ or parameter a_ρ is fixed (does not vary), the corresponding variation is equal to zero: $\delta z = 0$ ($z = t_0, t_1, x_i(t_0), x_i(t_1), a_\rho$). If any end condition $x_i(t_0)$, $x_i(t_1)$ or control parameter a_ρ is free the coefficient is equal to zero when there is free variation δz in (3.33)-(3.35).

Thus the set of conditions which express the maximum principle (3.26), (3.28), and conditions of transversality (3.29), give the necessary conditions of optimization of programmed control.

The conditions of the maximum principle enable us to distinguish those individual, generally speaking, isolated trajectories and controls, which can be optimal, on the set of all trajectories and controls which translate the system from (t_0, x_0) to (t_1, x_1).

Formulation of the maximum principle involves $2n + 2 + m$ unknown functions $x_0(t)$, $x_1(t)$, ..., $x_n(t)$: $\lambda_0(t)$, $\lambda_1(t)$, ..., $\lambda_n(t)$; $u_1(t)$, ..., $u_m(t)$, for determination of which there are $(n + 1)$ differential equations of physical system (3.14), (3.23), $(n + 1)$ differential equations of conjugate system (3.21) and m relations for u_j, derived from (3.27).

Consequently, for $2n + 2 + m$ unknown functions there are $2n + 2 + m$ relations. If all initial conditions are known /199

$$\left.\begin{aligned}\tilde{x}_0 = \tilde{x}(t_0) = (\Phi, x_1(t_0), x_2(t_0), \ldots, x_n(t_0))^T;\\ \lambda_0 = \lambda(t_0) = (\lambda_0(t_0), \lambda_1(t_0), \lambda_2(t_0), \ldots, \lambda_n(t_0))^T\end{aligned}\right\} \quad (3.36)$$

and the fixed value of the control parameter a is also known, system (3.26) can be integrated. However, the initial and final moments of time t_0, t_1, the initial and final values of the vector of phase coordinates $x_0 = (x_{10}, \ldots, x_{n0})$; $x_1 = (x_{11}, \ldots, x_{n1})$, the initial and final values of the vector of conjugate variables $\lambda_0 = (1, \lambda_{10}, \ldots, \lambda_{n0})$, $\lambda_1 = (1, \lambda_{11}, \ldots, \lambda_{n1})$, the constant vector $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ and the vector of control parameters $a = (a_1, a_2, \ldots, a_r)$ are known in advance for the optimal solution. They can be determined from conditions of transversality (3.31)-(3.35) and boundary conditions (3.15). Indeed, for determination of $(2 + 4n + l + r)$ unknowns t_0, t_1, x_0, x_1, λ_0, λ_1, μ, and a there are two conditions (3.31), (3.32), two terminal conditions (3.33), (3.34), r conditions (3.35) and l conditions (3.15); in addition, two terminal relations of the form $x(t_1) = \phi_1(t_0, t_1, \lambda_0, x_0)$, $\lambda(t_1) = \phi_2(t_0, t_1, \lambda_0, x_0)$ will be derived as the result of integration of system (3.26). Thus there is a sufficient number of relations to suggest that it is possible, at least theoretically, to solve the derived boundary problem. Numerical methods of solution of boundary problems are discussed in Chapter 4.

3.4.4. Certain Corollaries of the Maximum Principle

1. A direct corollary of system (3.26) and condition (3.27) is that function u(t) is satisfied between points of discontinuity in the relation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \quad (3.37)$$

This condition leads to the first integral for autonomous systems (i.e., systems which are not explicitly dependent on t): H = const on the entire optimal trajectory.

2. In most practical cases $\lambda_0 > 0$ (the so-called normal case) and therefore it is possible to assume $\lambda_0 = 1$ without loss of generality due to uniformity of function H for variables λ_i.

Comment

Due to the uniformity of H for λ_i, control u from (3.28) is determined not by the values of λ_i, but by their relations to one of them, for example, to λ_0. This is equivalent to the assumption $\lambda_0 = 1$. The case $\lambda_0 = 0$ is special (anormal) and is not examined here.

3. Conditions (3.27), (3.28) of the maximum principle make it possible to find optimal values for all m components of vector u.

If the minimum H for u is attained at an interior point of set U^m and functions f_i are differentiable for u, then f_j^* are determined from the condition

$$\left.\frac{\partial H}{\partial u_j}\right|_{u=u^*}=0 \quad (j=1, 2, \ldots, m). \tag{3.38}$$

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This condition, combined with (3.26), forms an Euler-Lagrange condition in the classical calculus of variations for problem (3.14), (3.15), (3.16) (see Section 3.9.2).

Comment

The minimum H for u is not always attained at an interior point of set U^m, but in such cases when it is attained at an interior point, the latter is not necessarily stationary (Figure 3.7). Types of minimizing points are quite varied (see Figure 3.7). Particularly noteworthy among them are cases of a nonrigorous minimum because the maximum principle does not enable one to determine u* uniquely for them. This is a special case in the theory of optimal control (see Section 3.6).

If function H achieves its minimum value at a point on the boundary $\Gamma_U m$ of region U^m, condition (3.38) is no longer necessary at this point. Here three cases are possible:

a) set U^m is described by a system of relations of the form of equalities

$$\chi_s(u_1, u_2, \ldots, u_m)=0 \quad (s=1, 2, \ldots, \nu<m); \tag{3.39}$$

and then the minimum H is found by the method of undefined Lagrange factors for conditions (3.39);

b) set U^m is given by system of inequalities /201

$$\varkappa_{s_1}(u_1, u_2, \ldots, u_m) \leqslant 0 \ (s_1 = 1, 2, 3, \ldots). \tag{3.40}$$

Then the problem reduces at each step of integration to a problem of nonlinear programming;

c) set U^m is a bounded region without boundaries (for example, a closed two-dimensional sphere or ellipsoid type surface in three-dimensional space). For each continuous function $H(u)$ which has continuous partial derivatives, given on a closed surface and expressed through parametric coordinates of this surface, the point of the maximum H for these parametric coordinates is one of the solutions of (3.38) where the part of u_j is played by the parametric coordinates of the surface.

Example. Let $H(u_1, u_2, u_3)$ be given on a sphere. Then substitution of $u_1 = r \sin\theta \cos\phi$, $u_2 = r \sin\theta \sin\phi$, $u_3 = r \cos\theta$ leads to $H(u_1, u_2, u_3) = H(\theta, \phi, r)$, a periodic function with period 2π for θ and ϕ, and at the point of a minimum $H = \tilde{H}$ the equalities

$$\frac{\partial \tilde{H}}{\partial \theta} = \frac{\partial \tilde{H}}{\partial \varphi} = 0.$$

4. Conditions (3.38) determine only the stationary point of function H. If $u = u^*$ satisfies system (3.38) and imparts minimum to function $H(u)$ then the necessary second-order conditions must be satisfied:

Matrix of second-order partial derivatives of function $H(u)$

$$H_{uu} = \left\{\frac{\partial^2 H}{\partial u_i \partial u_j}\right\} \ (i, j = 1, 2, \ldots, m) \tag{3.41}$$

must be non-negative-definite at point u^* of minimum of function $H(u)$.

The positive definiteness of matrix H_{uu} when conditions (3.38) are satisfied at point u^* is a sufficient condition for the relative (but not absolute!) minimum $H(u)$ at this point. Condition (3.41) of non-negative definiteness of matrix H_{uu} is the Legendre-Clebach condition of classical calculus of variations (see Section 3.9.4).

The positive definiteness of matrix H_{uu} can be checked with Sylvester's criterion:

for positive definiteness of matrix H_{uu} it is necessary and sufficient that its angular minors be positive.

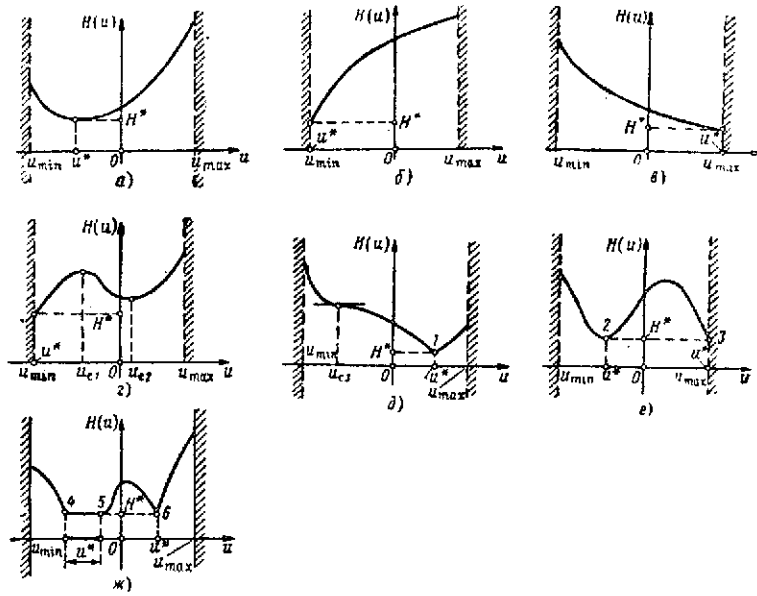

Figure 3.7. Examples of the Relationships of the Hamiltonian H to Control u and Types of Minimizing Points u^* in Set U:

$$u_{\min} \leqslant u \leqslant u_{\max}$$

a, Interior min $H(u)$ in stationary point; b, c, Boundary min $H(u)$; d, Boundary min $H(u)$; u_{s1}, u_{s2}, Stationary points of local max and min; e, Interior min $H(u)$ at angular point; u_{s3}, Point of inflection; f, Two isolated minimizing points 2 and 3; f, Non-rigorous min $H(u)$ on segment 4-5 and isolated min $H(u)$ at point 6.

In particular, for positive definite matrix H_{uu} the condition

$$\det \left\{ \frac{\partial^2 H}{\partial u_i \partial u_j} \right\} \Big|_{u^*} > 0, \tag{3.42}$$

is satisfied, which is the analog of Hilbert's condition of non-singularity (non-degeneracy) of a variational problem (see Section 3.9.4).

5. The formulation of the maximum principle given herein is also valid for the case when region U^m depends explicitly on time t: /202

$$U^m = U^m(t).$$

Comment.

The maximum principle is, generally speaking, only a necessary condition. Any permissible optimal control, if it exists, satisfies the maximum principle. However, not any permissible control, which satisfies the maximum principle, is optimal. Therefore, after determining a control on the basis of necessary conditions one should be convinced of its optimality. For this purpose there are sufficient conditions of optimality.

In certain cases the maximum principle is not only a necessary one, but also a sufficient condition of optimization of control u(t). Suppose, for example, that permissible control $u^*(t)$ is found, which translates the given initial state $x(t_0) = x_0$ of a system which is linear relative to phase coordinates

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{h}(\mathbf{u}, t), \quad \mathbf{u} \in U^m, \tag{3.43}$$

(where U^m is a closed bounded set; A(t), h(u, t) are continuous functions t, u; $x = (x_1, x_2, \ldots, x_n)$, $u = (u_1, u_2, \ldots, u_m)$) to a given terminal state $x(t_1) = x_1$. We introduce a system of initial conjugate variables

$$\boldsymbol{\lambda}(t_0) = (\lambda_{00}, \lambda_{10}, \ldots, \lambda_{n0})^T, \quad \lambda_{00} > 0,$$

such that $u^*(t)$ minimizes the function

$$H = \lambda_{00} h_0(\mathbf{u}, t) + \boldsymbol{\lambda}^T(t)\mathbf{h}(\mathbf{u}, t)$$

at each moment t for all $u \in U^m$, where

$$\dot{\boldsymbol{\lambda}}(t) = -A^T(t)\boldsymbol{\lambda}(t) - \lambda_{00}\frac{\partial f_0^T(\mathbf{x}^*(t), t)}{\partial \mathbf{x}}.$$

Then control $u^*(t)$ minimizes the quality criterion

$$J[\mathbf{u}(t)] = \int_{t_0}^{t_1} [f_0(\mathbf{x}, t) + h_0(\mathbf{u}, t)]\, dt,$$

on trajectories $x^*(t)$ of system (3.43), which pass through x_0, x_1, only if $f_0(x, t)$ is a unique downward convex[10] function of x for all $t \in [t_0, t_1]$.

---

[10] The function $f_0(x, t)$ is called convex downward in terms of x for $t \in [t_0, t_1]$ if for all $x \in R^n$, $\bar{x} \in R^n$

$$\frac{\partial f_0(x, t)}{\partial x}(\bar{x} - x) + f_0(x, t) < f_0(\bar{x}, t).$$

Example 1. Find the necessary conditions of optimality of programmed control of the thrust vector of a liquid-rocket engine (LRE) during guidance of a hypothetical winged aircraft to prescribed altitude at maximum end velocity (see [15]). /203

Mathematical model of guidance problem. Let the technical problem of aircraft guidance be adequately described by the mathematical model of the control process, formulated on the basis of the following assumptions.

1. The motion of an aircraft in guidance is sufficiently characterized by the equations of motion of its center of mass; the motion of the aircraft (as a solid) around its center of mass can be ignored; the change of mass of the aircraft during flight must be taken into account.

2. The aircraft flies above the spherical, stationary surface of the earth in the plane of a great circle. The Earth's gravitational field is assumed to be central Newtonian, i.e., gravitational acceleration is defined by the expression

$$g = g_0\left(\frac{R_e}{R_e+h}\right)^2,$$

where g_0 is the standard acceleration of gravity on the earth's surface; R_e is the radius of the earth and h is elevation above the earth's surface.

3. The aircraft flies under the influence of the thrust of its liquid fueled engine, aerodynamic forces and earth's attraction. The earth's atmosphere is characterized by known values of pressure p(h) and speed of sound a(h) as functions of altitude (by the parameters of the standard atmosphere).

4. The liquid fueled engine is rigidly fixed in the air frame. Motion in the guidance segment of the trajectory is controlled by rotating the longitudinal axis of the aircraft and by changing thrust.

Variables of State

In consideration of the above assumptions it is advantageous to use as the variables of state (phase coordinates) of the aircraft control process in the guidance segment, the following values, which (in accordance with the laws of mechanics) completely describe the plane motion of a point of variable mass:

$v(t)$ - is the flight velocity of the aircraft relative to the earth;
$\theta(t)$ - is the angle of inclination of the trajectory to the local horizon;
$h(t)$ - is altitude of flight above the earth's surface;
$L_e(t)$ - is the range of flight along earth's surface;
$m(t)$ - is the mass of the aircraft;
t - is time.

Thus phase vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T = (v, \theta, h, L_e, m)^T$ in the given problem consists of five components: $x_1 = v$; $x_2 = \theta$; $x_3 = h$; $x_4 = L_e$; $x_5 = m$.

Law of change of state of aircraft. Equations of motion. The above choice of variables of state is equivalent to analysis of aircraft motion in the guidance segment in velocity (aircraft-centric-Cartesian vertical-velocity) coordinate system C115. The equations of plane motion of the center of mass of the aircraft (as points of variable mass) in coordinate system C115 are

$$\left.\begin{aligned} \dot{v} &= \frac{P}{m}\cos\alpha - \frac{Q}{m} - g\sin\theta; \\ \dot{\theta} &= \frac{P}{mv}\sin\alpha + \frac{Y}{mv} - \left(\frac{g}{v} - \frac{v}{R_3 + h}\right)\cos\theta; \\ \dot{h} &= v\sin\theta;\quad \dot{L}_3 = \frac{vR_3}{R_3 + h}\cos\theta;\quad \dot{m} = -\mu_{sec}. \end{aligned}\right\} \qquad (I)$$

Here /204

$P = u_{fl}\mu_{sec}$ - the thrust of the liquid fueled engine, the axis of which is rigidly attached relative to the air frame and is directed along the wing chord;

u_{fl} - the mass flow rate;

μ_{sec} - fuel flow rate per second;

$Q = c_x qS$, $Y = c_y qS$ - aerodynamic drag and lift;

$q = 0.7\, p(h)M^2$ - the velocity head;

M - the Mach number;

c_x, c_y - aerodynamic coefficients;

S - the characteristic area of the aircraft;

α - the angle of attack.

Definition of the law which the evolution of state of the aircraft obeys during guidance [i.e., of equations of motion (I)] requires further definition of the aerodynamic forces acting upon it.

Let us assume that the aerodynamic coefficients are determined by relations that are valid for the elliptical polar

$$c_x = c_{x0}(M) + 2B_\alpha(M)[c_y^\alpha(M)]^2(1 - \cos\alpha) = \bar{c}_{x0}(M) - \bar{c}_x^l(M)\cos\alpha;$$

$$c_y = c_y^\alpha(M)\sin\alpha,$$

where $c_{x0}(M)$, $B_\alpha(M)$, $c_y^\alpha(M)$ - are known continuous and piece-wise smooth functions of M.

The system thus obtained is defined in the range $v > 0$, $m > 0$. It is autonomous, since it does not depend explicitly on t. We will note, moreover, that the right-hand parts of the system do not depend on L_e.

Control variables. Analysis of equation system (I) (after consideration of all functions entering the right parts) reveals that in addition to the chosen variables of state (v, θ, h, L_e, m), the values α and μ_{sec}, which are not yet defined, also enter in it. We will assume that α and μ_{sec} can change within known limits during the guidance process so as to have an effect on its characteristics. The assumption that α and μ_{sec} are some functions of time $\alpha = \alpha(t)$, $\mu_{sec} = \mu_{sec}(t)$, whose program of change can be chosen in accordance with our examination, corresponds to classification of α, μ_{sec} as control actions. Thus control vector u(t) in the given problem has two components

$$u(t) = (u_1(t), u_2(t)) = (\alpha(t), \mu_{sec}(t)).$$

The domain of the control variables is: $-\pi \leqslant \alpha \leqslant \pi$, $0 \leqslant \mu_{sec} < \infty$.

We will assume that the configuration of the aircraft is strictly defined and the flow rate u_{fl} is given. Then the problem contains no design (control) parameters. (If the possibility of change of μ_{sec} during the guidance process is denied, i.e., μ_{sec} = const, but the possibility of choice of μ_{sec} remains, then this value should be classified as a control parameter.)

Thus the controlled process of change of state of the aircraft during guidance obeys the following differential equations, written in the general symbols of Section 3.2:

$$\left.\begin{aligned}
&\dot{x}_1 = \frac{u_{st}}{x_5} u_2 \cos u_1 - \frac{Q(x_1, x_3, u_1)}{x_5} - g(x_3) \sin x_2 = \\
&\quad = f_1(x_1, x_2, x_3, x_5, u_1, u_2); \\
&\dot{x}_2 = \frac{u_{st}}{x_1 x_5} u_2 \sin u_1 - \frac{Y(x_1, x_3, u_1)}{x_5} - \left[\frac{g(x_3)}{x_1} - \right. \\
&\quad \left. - \frac{x_1}{R_3 + x_3}\right] \cos x_2 = f_2(x_1, x_2, x_3, x_5, u_1, u_2), \\
&\dot{x}_3 = x_1 \sin x_2 = f(x_1, x_2); \\
&\dot{x}_4 = \frac{x_1 R_3}{R_3 + x_3} \cos x_2 = f_4(x_1, x_2, x_3); \\
&\dot{x}_5 = -u_2 = f_5(u_2).
\end{aligned}\right\} \qquad \text{(II)}$$

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We will note that the functions $f_i(x)$ are continuous and piece-wise smooth functions within their domain. In applied problems, however, it is more convenient to use the symbols of the variables which indicate their physical

meaning. In the ensuing discussion, therefore, we will use equation system (I) in physical symbols of flight mechanics. Here we will assume that v, θ, h, L_e, m are phase coordinates and α, μ_{sec} are control variables:

$$\left.\begin{aligned}
&\dot{v} = \frac{u_{st}}{m}\mu_{sec}\cos\alpha - \bar{c}_{x0}(M)\frac{qS}{m} + \bar{c}_x^{\alpha}(M)\frac{qS}{m}\cos\alpha - \\
&\quad -g(h)\sin\theta; \\
&\dot{\theta} = \frac{u_{st}}{mv}\mu_{sec}\sin\alpha + c_y^{\alpha}(M)\frac{qS}{mv}\sin\alpha - \\
&\quad -\left[\frac{g(h)}{v} - \frac{v}{R_3 + h}\right]\cos\theta; \\
&\dot{h} = v\sin\theta; \\
&\dot{L}_3 = \frac{vR_3}{R_3 + h}\cos\theta; \\
&\dot{m} = -\mu_{sec}.
\end{aligned}\right\} \qquad \text{(III)}$$

Limitations on control actions and phase coordinates. We will assume that in accordance with the technical conditions of guidance, upon control variables α(t), $\mu_{sec}(t)$ are imposed the limitations

$$\left.\begin{aligned}
&\alpha_{min}(t) \leqslant \alpha(t) \leqslant \alpha_{max}(t); \\
&0 \leqslant \mu_{sec}(t) \leqslant \mu_{sec\,max},
\end{aligned}\right\} \qquad \text{(IV)}$$

where $\alpha_{min} < 0$; $\alpha_{max} > 0$; $\mu_{sec\ max}$ are known numbers.

There are no limitations on the phase coordinates (with the exception of $v > 0$, $m > 0$).

Classes of functions of control and phase variables. Before the mathematical tools of optimal processes theory, developed above, can be used reliably it must be determined to which class of time functions control variables α(t), $\mu_{sec}(t)$ correspond. Let the technical statement of the problem not preclude the assumption of piece-wise continuity of the functions α(t), $\mu_{sec}(t)$ (i.e., discontinuity of the function α(t), $\mu_{sec}(t)$ in a finite number of points has no effect on the physical feasibility of the control program). /206
Then piece-wise continuous functions α(t), $\mu_{sec}(t)$, satisfying limitations (IV) on the guidance interval, are called permissible.

In relation to the phase variables we will assume that they are continuous piece-wise smooth functions of time. In view of the continuity and piece-wise smoothness of the right sides of the differential equations the selected functional class of phase variables is consistent with the functional class of control actions. It should be pointed out, however, that the choice of class

of functions for the phase variables can be made without regard to the choice of class of control functions (thus, for example, mass m(t) could be a discontinuous function of time if concentrated loads are released or if stage seperation takes place. In this case the problem should be formulated as a problem with discontinuous phase coordinates, see Section 3.10). Thus any phase trajectories that satisfy system (III) for permissible control functions $\alpha(t)$, $\mu_{sec}(t)$ are permissible.

Initial and boundary conditions. At initial moment of time $t = t_0$ let the initial phase coordinates be given:

$$v(t_0) = v_0; \quad \theta(t_0) = \theta_0; \quad h(t_0) = h_0; \quad L_3(t_0) = L_{30}; \quad m(t_0) = m_0. \tag{V}$$

Since system (III) is autonomous and its right sides do not depend on L_e, then it may be assumed that $t_0 = 0$, $L_{e0} = 0$. Then at the final moment of guidance $t = t_1$ the boundary values of the following phase coordinates are given:

$$\theta(t_1) = \theta_1; \quad h(t_1) = h_1; \quad m(t_1) = m_1. \tag{VI}$$

Here t_1 and $L_{e1} = L_e(t_1)$ are not fixed (are free). This type of boundary conditions corresponds to the process of insertion of the aircraft into a given terminal state θ_1, h_1, m_1, during which process insertion time t_1 and range of insertion L_{e1} are technically unimportant.

Aircraft guidance quality criterion. Optimality criterion. We will evaluate the various means of guidance [i.e., control programs $\alpha(t)$, $\mu_{sec}(t)$] of an aircraft into prescribed conditions in terms of velocity $v(t_1)$ at the end of process (VI) of insertion into the prescribed conditions. The function $v(t_1)$ in this statement is a functional of $u = (\alpha(t), \mu_{sec}(t))$ and a function of the free insertion time t_1 and insertion range. We will assume the aircraft control process to be optimal if control variables $\alpha(t)$, $\mu_{sec}(t)$ and final values of t_1, L_{e1} are selected such that they maximize terminal velocity $v(t_1)$. Thus the quality criterion J in this problem is

$$J[t_1, L_{31}, (\alpha(t), \mu_{sec}(t))] = v(t_1) \tag{VII}$$

and the problem of optimization of the guidance process consists in determining the value

$$J^* = \max_{t_1, L_{31}, \alpha(t), \mu_{sec}(t)} v(t_1). \tag{VII'}$$

Exact formulation of optimization problems. Of all permissible control functions $u = (\alpha(t), \mu_{sec}(t))$ and all t_1, $L_{el} = L_e(t_1)$, find those for which the aircraft is transferred from given initial state (V) to partially given final state (VI) and $J = v(t_1)$ acquires the maximum value $J^* = J_{max}$. Thus

$$J^* = \max_{t_1,\, L_{31}} \max_{\alpha(t),\, \mu_{sec}(t)} J[t_1, L_{31}, \alpha(t), \mu_{sec}(t)] = \max_{t_1, L_{31}} \max_{\alpha(t),\, \mu_{sec}(t)} v(t_1); \qquad /207$$

$$(t_1^*, L_{31}^*, \alpha^*(t), \mu_{sec}^*(t)) = \arg \max_{t_1,\, L_{31}} \max_{\alpha,\, \mu_{sec}} J[t_1, L_{31}, \alpha(t), \mu_{sec}(t)]$$

(the asterisk denotes the optimal t_1, L_{el} and optimal control programs $\alpha(t)$, $\mu_{sec}(t)$).

Necessary conditions of optimality of control $\alpha(t)$, $\mu_{sec}(t)$ and final t_1, L_{el}. To determine the necessary conditions of optimality of control $\alpha(t)$, $\mu_{sec}(t)$ we will use the maximum principle, and to find the necessary conditions of optimality of t_1 and L_{el}, the conditions of transversality (see Section 3.4.3). For this purpose we introduce the vector $\lambda = (\lambda_v, \lambda_\theta, \lambda_h, \lambda_L, \lambda_m)^T$ of conjugate variables and write the Hamiltonian H of system (III):

$$H = \lambda_v \left[\frac{u_{st}\mu_{sec}}{m} \cos\alpha - \frac{Q(v, h, \alpha)}{m} - g(h)\sin\theta \right] +$$
$$+ \lambda_\theta \left[\frac{u_{st}\mu_{sec}}{mv} \sin\alpha + \frac{Y(v, h, \alpha)}{mv} - \left(\frac{g(h)}{v} + \frac{v}{R_3 + h} \right) \cos\theta \right] +$$
$$+ \lambda_h v \sin\theta + \lambda_L \frac{R_3 v}{R_3 + h} \cos\theta - \lambda_m \mu_{sec} = \bar{H}(\alpha, \mu_{sec}) + H_0, \qquad \text{(VIII)}$$

where

$$\bar{H}(\alpha, \mu_{sec}) = \left(\lambda_v \frac{u_{st}}{m} \cos\alpha + \lambda_\theta \frac{u_{st}}{mv} \sin\alpha - \lambda_m \right) \mu_{sec} +$$
$$+ \lambda_v \bar{c}_x^{\,l}(v, h) \frac{qS}{m} \cos\alpha + \lambda_\theta c_y^{\alpha}(v, h) \frac{qS}{mv} \sin\alpha$$

is the "controlled" (i.e., dependent on control variables α, μ_{sec}) part of Hamiltonian H;

$$H_0 = -\lambda_v \left[g(h)\sin\theta + \bar{c}_{x0}(v, h) \frac{qS}{m} \right] - \lambda_\theta \left[\frac{g(h)}{v} - \frac{v}{R_3 + h} \right] \cos\theta +$$
$$+ \lambda_h v \sin\theta + \lambda_L \frac{R_3 v}{R_3 + h} \cos\theta$$

is the "uncontrolled" (i.e., not dependent on control variables) part of Hamiltonian H.

$\tilde{H}(\alpha, \mu_{sec})$, can be written as

$$\tilde{H}(\alpha, \mu_{sec}) = F_1(\alpha)\mu_{sec} + F_2(\alpha),$$

where

$$F_1(\alpha) = \lambda_v \frac{u_{st}}{m}\cos\alpha + \lambda_\theta \frac{u_{st}}{mv}\sin\alpha - \lambda_m;$$

$$F_2(\alpha) = 2\lambda_v \frac{qS}{m} B_a \left(c_y^\alpha\right)^2 \cos\alpha + \lambda_\theta c_y^\alpha \frac{qS}{mv}\sin\theta.$$

The equation system conjugate to the initial system (III), with the aid of function H is written as:

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$$\left.\begin{aligned}
\dot{\lambda}_v &= -\frac{\partial H}{\partial v} = \frac{\lambda_v}{m}\frac{\partial Q}{\partial v} + \lambda_v \frac{\mu_{sec} u_{st}}{mv^2}\sin\alpha - \frac{\lambda_\theta}{m}\frac{\partial}{\partial v}\left(\frac{Y}{v}\right) - \\
&\quad - \frac{\lambda_\theta}{v^2} g\cos\theta - \frac{\lambda_\theta \cos\theta}{R_3 + h} - \lambda_h \sin\theta - \lambda_L \frac{R_3}{R_3 + h}\cos\theta; \\
\dot{\lambda}_\theta &= -\frac{\partial H}{\partial \theta} = \lambda_v g\cos\theta - \lambda_\theta\left(g\sin\theta - \frac{v}{R_3 + h}\sin\theta\right) - \\
&\quad - \lambda_h v\cos\theta + \lambda_L \frac{R_3 v}{R_3 + h}\sin\theta; \\
\dot{\lambda}_h &= -\frac{\partial H}{\partial h} = \frac{\lambda_v}{m}\frac{\partial Q}{\partial h} + \lambda_v \frac{\partial g}{\partial h}\sin\theta - \lambda_\theta\left(\frac{1}{mv}\frac{\partial Y}{\partial h} - \right. \\
&\quad \left. - \frac{1}{v}\frac{\partial g}{\partial h}\cos\theta - \frac{v\cos\theta}{(R_3 + h)^2}\right) + \lambda_L \frac{R_3 v}{(R_3 + h)^2}\cos\theta; \\
\dot{\lambda}_m &= -\frac{\partial H}{\partial m} = \lambda_v\left(\frac{u_{st}\mu_{sec}}{m^2}\cos\alpha - \frac{Q}{m^2}\right) + \\
&\quad + \lambda_\theta\left(\frac{v_{st}\mu_{sec}}{m^2 v}\sin\alpha + \frac{Y}{m^2 v}\right); \\
\dot{\lambda}_L &= -\frac{\partial H}{\partial L_3} = 0.
\end{aligned}\right\} \qquad \text{(IX)}$$

The maximum Hamiltonian H in terms of variables α, μ_{sec}, which satisfy conditions (IV), is attained at points α^* and μ^*_{sec}, which imparts the maximum value to the controlled part $\tilde{H}(\alpha, \mu_{sec})$:

$$(\alpha^*, \mu^*_{sec}) = \underset{\substack{\alpha_{min}\le\alpha\le\alpha_{max} \\ 0\le\mu_{sec}\le\mu_{sec\,max}}}{\arg\max} \tilde{H}(\alpha, \mu_{cek}) =$$

$$= \underset{\substack{\alpha_{min}\le\alpha\le\alpha_{max} \\ 0\le\mu_{sec}\le\mu_{sec\,max}}}{\arg\max} \{F_1(\alpha)\mu_{sec} + F_2(\alpha)\}.$$

The problem of maximization of $\tilde{H}(\alpha, \mu_{sec})$ on rectangle ($\alpha_{min} \leqslant \alpha \leqslant \alpha_{max}$, $0 \leqslant \mu_{sec} \leqslant \mu_{sec\ max}$) pertains to problems of nonlinear programming. In this case the solution can be found from the following considerations. Since for all fixed $\alpha = \bar{\alpha}$ the value $\tilde{H}(\alpha, \mu_{sec})$ becomes maximum in terms of μ_{sec} for the values

$$\bar{\mu}_{sec} = \begin{cases} \mu_{sec\ max}, & \text{if } F_1(\bar{\alpha}) > 0; \\ 0 & \text{if } F_1(\bar{\alpha}) < 0; \\ \text{Any } \mu_{sec} & \text{of } 0 \leqslant \mu_{sec} \leqslant \mu_{sec\ max}, \\ & \text{if } F_1(\bar{\alpha}) \equiv 0, \end{cases} \tag{X}$$

then the optimal angles of attack α^* corresponding to these $\bar{\mu}_{sec}$ are found from the conditions:

1. For $\bar{\mu}_{sec} = 0$ (or $F_1(\alpha) = 0$)

$$= \begin{cases} \alpha_{1s} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{min} \leqslant \alpha_{1s} \leqslant \alpha_{max}; \\ \alpha_{max} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{1s} \geqslant \alpha_{max} \text{ or} \\ & \quad \lambda_v \leqslant 0 \ \& \ \alpha_{1s} \leqslant \frac{1}{2}(\alpha_{min} + \alpha_{max}); \\ \alpha_{min} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{1s} \leqslant \alpha_{min} \text{ or} \\ & \text{for } \lambda_v \leqslant 0 \ \& \ \alpha_{1s} \geqslant \frac{1}{2}(\alpha_{min} + \alpha_{max}), \end{cases} \tag{XI}$$

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where α_{1s} is the stationary point of function $\tilde{H}(\alpha, 0) = F_2(\alpha)$, determined from the equation

$$\frac{\partial \tilde{H}(\alpha, 0)}{\partial \alpha} = \frac{\partial F_2(\alpha)}{\partial \alpha} = -2\lambda_v B_a c_y^a \sin\alpha + \lambda_\theta \frac{\cos\theta}{v} = 0.$$

Hence,

$$\alpha_{1s} = \arctan \frac{\lambda_\theta}{2\lambda_v v B_a c_y^a}.$$

Comment. Equations (XI) are derived with consideration of the main inverse trigonometric function arctan x.

2. For $\overline{\mu}_{sec} = \mu_{sec\ max}$

$$\alpha^* = \alpha_2 = \arg\max_{\alpha} \tilde{H}(\alpha, \mu_{sec_{max}}) =$$

$$= \begin{cases} \alpha_{2S} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{min} \leqslant \alpha_{2S} \leqslant \alpha_{max}; \\ \alpha_{max} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{2S} \geqslant \alpha_{max} \text{ or} \\ & \text{for } \lambda_v \leqslant 0 \ \& \ \alpha_{2S} \leqslant \frac{1}{2}(\alpha_{min} + \alpha_{max}); \\ \alpha_{min} & \text{for } \lambda_v \geqslant 0 \ \& \ \alpha_{2S} \leqslant \alpha_{min} \text{ or} \\ & \text{for } \lambda_v \leqslant 0 \ \& \ \alpha_{2S} \geqslant \frac{1}{2}(\alpha_{min} + \alpha_{max}), \end{cases}$$

where α_{2s} is the stationary point of function $\tilde{H}(\alpha, \mu_{sec\ max})$, determined from the condition

$$\frac{\partial \tilde{H}(\alpha, \mu_{sec_{max}})}{\partial \alpha} = \frac{\partial F_1(\alpha)}{\partial \alpha} \mu_{sec_{max}} + \frac{\partial F_2(\alpha)}{\partial \alpha} = 0.$$

Hence (for $F_1(\alpha) \neq 0$)

$$\alpha_{2S} = \arctan\left\{\frac{1}{v}\frac{\lambda_\theta}{\lambda_v}\frac{a_{st}\mu_{sec_{max}} + c_y^* qS}{v_{st\ sec\ max} + 2B_\alpha (c_y^*)^2 qS}\right\}.$$

Now the optimal α^*, μ^*_{sec} can be found from the following relations (for $F_1(\alpha) \neq 0$):

$$\alpha^* = \begin{cases} \alpha_1, & \text{if } \tilde{H}(\alpha_1, 0) > \tilde{H}(\alpha_2, \mu_{sec_{max}}); \\ \alpha_2, & \text{if } \tilde{H}(\alpha_1, 0) < \tilde{H}(\alpha_2, \mu_{sec_{max}}); \\ \text{not defined}, & \text{if } \tilde{H}(\alpha_1, 0) = \tilde{H}(\alpha_2, \mu_{sec_{max}}); \end{cases}$$

$$\mu^*_{sec} = \begin{cases} 0, & \text{if } \tilde{H}(\alpha_1, 0) > \tilde{H}(\alpha_2, \mu_{sec_{max}}); \\ \mu_{max}, & \text{if } \tilde{H}(\alpha_1, 0) < \tilde{H}(\alpha_2, \mu_{sec_{max}}); \\ \text{not defined}, & \text{if } \tilde{H}(\alpha_1, 0) = \tilde{H}(\alpha_2, \mu_{sec_{max}}). \end{cases}$$

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For $F_1(\alpha) \equiv 0$ the so-called special mode of optimal control in terms of μ_{sec} (see section 3.6) is valid. In this case the optimal control of the angle of attack is found from the condition

$$\alpha^* = \alpha_1,$$

and optimal control of fuel flow rate μ_{sec} cannot be determined with the aid of the maximum principle. The special mode was not discussed in detail in this example (see Section 3.6 concerning methods of analyzing special modes).

For $\tilde{H}(\alpha_1, 0) = \tilde{H}(\alpha_2, \mu_{sec\ max})$ and $F_1(\alpha) \neq 0$ the point at which function $\tilde{H}(\alpha, \mu_{sec})$ attains its absolute maximum in terms of α, μ_{sec} on the set (IV) is

not unique. If this equality is observed only at isolated moments of time, then it doesn't matter which pair of values of $(\alpha_1, 0)$ and $(\alpha_2, \mu_{sec\ max})$ is chosen. If, however, this equality is satisfied on some interval of time, then the so-called "sliding" mode may occur.

To find the necessary conditions of optimality of free (unknown) t_1 and $L_{e1}=L_e(t_1)$ we will write the conditions of transversality. The general conditions of transversality in the problem at hand acquire the form

$$H\delta t - \lambda_v \delta v - \lambda_\theta \delta\theta - \lambda_h \delta h - \lambda_L \delta L_3 - \lambda_m \delta m \Big|_{t_0}^{t_1} + \delta v_1 = 0.$$

Since the left end is fixed, then $\delta t_0 = \delta v_0 = \delta\theta_0 = \delta h_0 + \delta L_{e0} = \delta m_0 = 0$. Furthermore, since the coordinates θ_1, m_1, h_1 on the right end are known, then $\delta\theta_1 = \delta h_1 = \delta m_1 = 0$.

Thus, finally:

$$H(t_1)\delta t_1 - \lambda_v(t_1)\delta v_1 - \lambda_L(t_1)\delta L_{31} + \delta v_1 = 0.$$

Hence, in view of the arbitrary nature of variations δt_1, δv_1, δL_{e1} we obtain

$$H(t_1) = 0;\ \lambda_v(t_1) = 1;\ \lambda_L(t_1) = 0. \tag{XII}$$

It follows from the equation for λ_L (see (IX)) that λ_L = const; then from (XII) we obtain

$$\lambda_L \equiv 0.$$

The maximum principle establishes that the Hamiltonian H and conjugate variables λ_i are continuous on the optimal trajectory. Since the initial equation system (I) is autonomous, it then follows from the equality $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$ (corollary of the maximum principle) and continuity of Hamiltonian H on the optimal trajectory, that the problem has the first integral

$$H(t) = \text{const} = C,$$

whereupon, in view of the conditions of transversality on the optimal trajectories $C = 0$.

The set of conditions thus obtained is "sufficient" (in terms of Section 3.4.3) for the numerical solution of the boundary value problem (see Chapter 4). We should note, however, that the conditions of optimality of α, μ_{sec}, t_1 L_{e1} thus

obtained, are only necessary conditions. Therefore not every solution of the boundary value problem will yield optimal control.

3.5. Necessary Conditions of Optimality for Basic Problem of Control Law Synthesis. Dynamic Programming Method. /211

3.5.1. Problem of Synthesis of Optimal Control Law

It is more natural to use for synthesizing the optimal law of control systems with feedback, optimal closed control systems, optimal guidance laws, etc. (see Section 3.3.2) a method other than the one employed for the solution of problems described in Sections 3.4 and 3.9.

In contrast to the Euler-Lagrange equations and Pontryagin's maximum principle, which incorporate temporal representation of optimal control [in the form $u^* = u(t)$] for a single controlled process, this approach examines optimal control as the law $u^* = v^*(x, t)$ (coordinate control, feedback control) or a set of homogeneous, identical processes with different initial states.

From the standpoint of mechanics this approach corresponds to examination of the propagation of "perturbation waves" from some source in a heterogeneous atmosphere. The generality of both methods is established by design geometry, from which standpoint the trajectory of a point in phase space can be regarded as a series of points and as the envelope of their tangents.

The subsequent use of the described method for problems of optimal control produces for continuous processes a differential equation (nonlinear) in partial first-order derivatives of the Hamilton-Jacobi type.

One possible way to find this equation is to use the dynamic programming optimality principle. Dynamic programming is a rather general method, developed for the solution of general multi-stage choice problems (i.e., for problems in which the results of preceding operations can be used for controlling the course of future operations).

3.5.2. Principle of Dynamic Programming Optimality

Principle of optimality. Dynamic programming is the foundation upon which R. Bellman formulated the optimality principle. The basic property of an optimal trajectory (optimal control), stated in Section 3.3.3, can be formulated as the following optimality principle.

Optimal control is independent of how the system arrives at a given state for $t = t'$ (i.e., does not depend on the "history" of motion), and for future moments in time is completely defined only by the state of the system at the examined moment of time. /212

Problems of control of continuous processes (of the type formulated in Section 3.3.2 for the basic problem of optimal coordinate control) are regarded as a special case of dynamic programming.

Brief Formulation of problem. Let the system of motion equations be given:

$$\frac{dx}{dt}=f(t,\ x,\ u), \tag{3.44}$$

where $u = (u_1, u_2, \ldots, u_m)^T \in U^m$;

$$=(u_1, u_2, \ldots, u_m)^T \in U^m;$$

$$x=(x_1, x_2, \ldots, x_n)^T \in X^n;$$

$$f=(f_1(t, x, u), f_2(t, x, u), \ldots, f_n(t, x, u))^T;$$

and the boundary conditions are

$$x(t_0)=x_0;\quad x(t_1)=x_1. \tag{3.45}$$

It is required to synthesize the law of optimal control $u^* = v^*(x, t)$ which minimizes the functional

$$J[t_0, x_0, u]=\int_{t_0}^{t_1} f_0(t, x, u)dt. \tag{3.46}$$

Necessary conditions. Suppose in $(n + 1)$-dimensional space (X^n, T) there exists some region $G(x, t)$ with initial values x_0, t_0 $((x_0, t_0) \in G(x, t))$, for each point of which there exists optimal (in terms of minimum $J[t_0, x_0, u]$) control $u^*(t)$, which translates these initial points to some fixed point $(x(t_1) = x_1, t_1)$; x_1, t_1 are given. In such optimal controls the minimal value of quality criterion (3.46) will only depend on the initial values of x_0, t_0. Thus

$$J_{min}=J^*=V(t_0, x_0),$$

where $V(t_0, x_0)$ is some function of the $(n + 1)$ variables t_0, x_{10}, $\ldots$, x_{n0}.

Bearing in mind an arbitrary point in region $G(x, t)$, for the purpose of simplifying the notation, the subscript "0" will be omitted.

Thus the function $V(t, x)$ is the minimal quality criterion (3.46) in the optimal trajectories of system (3.44), which originate at point (t, x) and end at fixed point (t_1, x_1):

$$V(t, x)=\min_{u\in U^m}\int_{t}^{t_1} f_0(t, x, u)dt \tag{3.47}$$

on trajectories (3.44) from (t, x) to (t, x_1).

Comment. The function $V(t, x)$ is the analog of "action" in analytical mechanics and of the "extremum integral" in classical calculus of variations. /213

If the function V(t, x) exists and is continuously differentiable for (t, x), it satisfies the basic equation of dynamic programming, or the first-order partial differential equation (the Hamilton-Bellman equation):

$$\frac{\partial V}{\partial t}+\min_{u\in U^m} H\left(t,\ x,\ \frac{\partial V}{\partial x},\ u\right)=0, \tag{3.48}$$

with the boundary conditions

$$V(t_1,\ x_1)=0; \tag{3.49}$$

Here

$$H(t,\ x,\ V_x,\ u)=f_0(t,\ x,\ u)+V_x f(t,\ x,\ u), \tag{3.50}$$

where $V_x=\frac{\partial V}{\partial x}$

(see Table 3.2)

Comment. Equation (3.48) is analogous to the Hamilton-Jacobi equation of classical calculus of variations:

$$\frac{\partial V}{\partial t}+H\left(t,\ x,\ \frac{\partial V}{\partial x}\right)=0, \tag{3.51}$$

where the function H is derived as the result of substituting into function $H(t,\ x,\ V_x,\ u)$ the control $u^0 = u^0(t,\ x,\ V_x)$, which is found from the condition of stability of this function:

$$\frac{\partial H}{\partial u_j}=0 \qquad (j=1,\ 2,\ \ldots,\ m). \tag{3.52}$$

From (3.48) it is possible to determine the optimal law of control:

$$u^*=v^*(t,\ x)=\operatorname*{argmin}_{u\in U^m} H\left(t,\ x,\ \frac{\partial V}{\partial x},\ u\right)=u^*\left(t,\ x,\ \frac{\partial V}{\partial x}\right). \tag{3.53}$$

The geometric sense of condition (3.53) is explained in Figure 3.8. If the function V(t, x) is found by means of solving equation (3.48) with condition (3.49) the problem of synthesis is solved because for the known function V(t, x) we have

$$u^*=u^*\left(t,\ x,\ \frac{\partial V(t,\ x)}{\partial x}\right)=v^*(t,\ x). \tag{3.54}$$

Just as Pontryagin's maximum principle simplifies and refines the Weierstrass condition (see Section 3.9.3) for the basic problem of optimal programmed control for the case of a closed region of values of control U^m, so also the Hamilton-Bellman equation is a refinement and generalization of the /214

Hamilton-Jacobi equation. The refinement consists in that in place of the condition of stability $\partial H/\partial u = 0$ where it does not correspond to reality, (3.48) uses the condition

$$\min_{u \in U^m} H\left(t,\ x,\ \frac{\partial V}{\partial x},\ u\right).$$

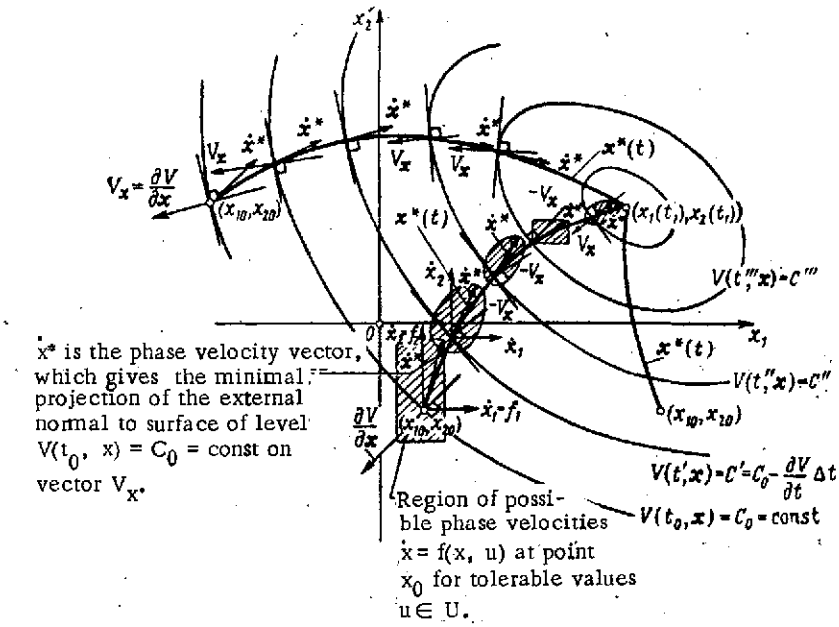
x* is the phase velocity vector, which gives the minimal projection of the external normal to surface of level V(t0, x) = C0 = const on vector Vx.
Region of possible phase velocities ẋ = f(x, u) at point x0 for tolerable values u ∈ U.
V(t0, x) = C0 = const

Figure 3.8. Geometric Meaning of Condition $\min H(t, x, V_x, u) = \min_{u \in U^m} [V_x f(t, x, u)]$:

$$V(t, x) = \min_{u \in U^m} J[u(t)],\ V_x = \frac{\partial V}{\partial x},\ n = m = 2,\ f_0 = 0,$$

x* is optimal phase velocity: x* = f(t, x, u*);

u*(t, x) is optimal control: $u^* = \arg\min_{u \in U^m} H(t, x, V_x, u)$;

x* is optimal trajectory.

In given condition (3.48) the requirement of continuous differentiability (smoothness) of function V(t, x) is important. But in contrast to the maximum principle, where the existence of the vector-function $\lambda(t)$ necessary for it is confirmed, the existence of a smooth potential V(t, x) is not proved in the method of dynamic programming. This detracts from the value of necessary condition (3.48) because for the non-smooth function V(t, x) it is difficult to maintain its indispensibility to the fullest extent. /215

3.5.3. Relaxed Necessary Condition

The exact necessary condition for the basic problem of optimal coordinate control based on the optimization principle, which is partially relieved from the requirement of continuous differentiability of the function V(t, x), is formulated as follows.

Formulation of the problem. Let the boundary condition be

$$x(t_0)=x_0;\quad q(t_1,\ x(t_1))=0. \tag{3.55}$$

The minimized functional is

$$J[t_0,\ x_0,\ u]=\Phi(t_1,\ x(t_1))+\int_{t_0}^{t_1} f_0(t,\ x,\ u)dt \tag{3.56}$$

and is determined on trajectories of system (3.44) with control

$$u(t)\in U^m(t,\ x).$$

Law of control v(t, x) is considered tolerable if u(t) = v(t, x(t)), $v(t,\ x(t))\in U^m(t,\ x)$ and is piece-wise continuous.

If control $u = u^*(t)$, $t_0 \leq t \leq t_1$ imparts the minimum to functional J the optimal trajectory x*(t) corresponds to it.

Let

$$V(t_0,\ x_0)=\min_{u\in U^m}\left\{\Phi(t_1,\ x(t_1))+\int_{t_0}^{t_1} f_0(t,\ x,\ u)dt\right\}=$$

$$=\Phi(t_1,\ x^*(t_1^*))+\int_{t_0}^{t_1^*} f_0(t,\ x^*(t),\ u^*(t))dt. \tag{3.57}$$

Then

$$V(t_0,\ x_0)\leqslant\Phi(t_1,\ x(t_1))+\int_{t_0}^{t_1} f_0(t,\ x(t),\ u(t))dt,$$

where u(t) is arbitrary.

Necessary conditions. It is assumed that unknown optimal control u* = = v*(t, x) exists. Then it is possible to establish the necessary conditions for the basic problem of optimal coordinate control.

Let the following conditions be satisfied in the region G of space of states X^n. /216

1. For $x \in G$ at moment t the function

$$H\left(t, x, \frac{\partial V}{\partial x}, u\right) = f_0(t, x, u) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x, u)$$

has an absolute minimum for u, i.e., $\min_u H = H^*(t, x, V_x)$ when $u^* = v^*(t, x) = u^*(t, x, V_x)$ for all tolerable $u(t) \in U^m(t, x)$, where $V_x = \partial V/\partial x$ is the gradient $V(t, x)$.

2. The solution of $x(t)$ of system (3.44) exists and is a continuous function for all tolerable $u(t) \in U^m(t, x)$.

3. The function $f_0(t, x, u)$ is continuous for t.

4. The function $V_t(t, x) = \partial V/\partial t$ is continuous for t and x; the vector-functions $V_x(t, x)$ and $f(t, x, u)$ are either continuous for t and x or have equal left and right limits for inner product $V_x \cdot f$ on any trajectory $x(t)$ of system (3.44):

$$\lim_{t \to t_0 + 0} [V_x(t, x) f(t, x(t)), u(t))] = \lim_{t \to t_0 - 0} [V_x(t, x) f(t, x(t), u(t))].$$

5. Optimal motion exists for each initial $x_0 \in G$ to some state which satisfies condition $q(t_1, x_1) = 0$ and such that the trajectory does not leave G.

6. Each point in G, which does not satisfy condition $q(t, x) = 0$, has a vicinity lying entirely within G.

Then function $V(t, x)$ in region G satisfies the Hamilton-Bellman equation

$$\min_{u \in U^m} \left\{ \left[\frac{dV}{dt} \right]_u + f_0(t, x(t), u(t)) \right\} = 0, \tag{3.58}$$

or

$$\min_{u \in U^m} \left\{ \frac{\partial V(t, x)}{\partial t} + V_x(t, x) f(t, x, u) + f_0(t, x, u) \right\} =$$

$$= \frac{\partial V(t, x)}{\partial t} + \min_{u \in U^m} H(t, x, V_x(t, x), u) =$$

$$= \frac{\partial V(t, x)}{\partial t} + H^*(t, x, V_x(t, x)) = 0 \tag{3.58'}$$

with the boundary condition

$$V(t, x) = \Phi(t, x) \tag{3.58''}$$

on the hypersurface $q(t, x) = 0$.

Here is defined:

$$H^*(t, \mathbf{x}, V_{\mathbf{x}}(t, \mathbf{x})) = \min_{\mathbf{u} \in U^m} H(t, \mathbf{x}, V_{\mathbf{x}}(t, \mathbf{x}), \mathbf{u}),$$

$[\frac{dV}{dt}]_u$ is the complete derivative on the trajectory achieved under the influence of control u.

Since for the known function V(t, x)

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$$\mathbf{u}^* = \arg\min_{\mathbf{u} \in U^m} H = \mathbf{u}^*(t, \mathbf{x}, V_{\mathbf{x}}(t, \mathbf{x})) = \mathbf{v}^*(t, \mathbf{x}),$$

then the solution V(t, x) found for equation (3.58) simultaneously provides the solution to the problem of synthesis of the optimal law of control.

Comments. 1. Requirement 4 entails continuity of functions $[dV/dt]_u$ and V(t, x) in time t.

2. When V_t, V_x and f_i are continuous with respect to t and x, equation (3.58) is the Hamilton-Jacobi equation.

The general order of operations which should be used for solving of the problem of synthesis of the optimal control law by the method of dynamic programming, is given in Table 3.2.

TABLE 3.2. SUMMARY OF GENERAL PROCEDURES OF METHOD OF DYNAMIC PROGRAMMING FOR CALCULATION OF OPTIMAL CONTROL LAW u* = v*(t, x)

Step 1	Function H is formed, in which conjugate variables λ_i are replaced by components of the vector $$\frac{\partial V}{\partial \mathbf{x}} = \operatorname{grad}_x V(t, \mathbf{x}) = V_{\mathbf{x}} = \left(\frac{\partial V(t, \mathbf{x})}{\partial x_1}, \frac{\partial V(t, \mathbf{x})}{\partial x_2}, \ldots \frac{\partial V(t, \mathbf{x})}{\partial x_n}\right),$$ i.e., $$H(t, \mathbf{x}, \mathbf{u}, V_{\mathbf{x}}) = V_{\mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + f_0(t, \mathbf{x}, \mathbf{u})$$
Step 2	$H(t, x, u, V_x)$ is minimized for $u \in U^m$ and the explicit dependence of control u* on components of the vector V_x is found: $$\mathbf{u}^* = \mathbf{u}^*(\mathbf{x}, V_{\mathbf{x}}, t) = \arg\min_{\mathbf{u} \in U^m} H(t, \mathbf{x}, \mathbf{u}, V_{\mathbf{x}})$$

Table 3.2. (Continued)

Step 3	The minimum H^* is found by substituting for H the values $u^*(t, x, V_x)$: $$H^*(t, x, V_x) = H(t, x, u^*(t, x, V_x), V_x)$$
Step 4	The Hamilton-Bellman partial differential equation is solved $$H^*(t, x, V_x) + \frac{\partial V}{\partial t} = 0$$ with the corresponding boundary condition for the function $V(t, x)$: $V(t, x) = \Phi(t, x)$ on the hypersurface $q(t, x) = 0$
Step 5	By substituting the results of Step 4 into the expression for $u^*(t, x, V_x)$, the law of control with feedback is found: $$u^* = v^*(t, x) = u^*\left(t, x, \frac{\partial V(t, x)}{\partial x}\right)$$

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Example 2. Synthesis of an optimal law of control for linear system with quadratic quality criterion. Problem of analytical construction of optimal autopilots.

Let a nonstationary linear system be described by a vectorial linear differential equation

$$\dot{x} = A(t)x + B(t)u + Cf(t) \tag{I}$$

with the initial condition

$$x(t_0) = x; \quad t_0 \leqslant t \leqslant t_1, \tag{II}$$

where t_1 is fixed, t_0, x_0 are known values (which, however, are not specially selected), and let the quality criterion be

$$J[u] = l_1^T x_1 + \frac{1}{2} x_1^T R_1 x_1 + \int_{t_0}^{t_1} [l_2^T(t)x(t) + l_3^T(t)u + \frac{1}{2}(x^T Q(t)x + x^T N(t)u + u^T N^T(t)x + u^T P(t)u)]dt. \tag{III}$$

Here

$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$; $\mathbf{f} = (f_1, \ldots, f_n)^T$; C, $A(t)$ are matrices of dimensionality $n \times n$; $\mathbf{u} = (u_1, \ldots, u_m)^T$, $\mathbf{x}_1 = \mathbf{x}(t_1)$;

B(t), N(t) are matrices of dimensionality n × m; R_1, Q(t) are positive semi-definite[11] symmetrical matrices of dimensionality n × n; P(t) is a positive definite symmetrical matrix of dimensionality m × m; f(t) is a known time function l_1, $l_2(t)$ are n-dimensional vectors, $l_3(t)$ is an m-dimensional vector. /219

It is assumed that no limitations are imposed on the control vector u, and matrices Q(t), N(t), P(t) are such that the condition is satisfied

$$Q(t) - N(t)P^{-1}(t)N^T(t) \geqslant 0$$

[This condition guarantees the absence of the conjugate (see Section 3.9.5) points in the given problem].

It is necessary to find the law of control with feedback

$$\mathbf{u}^* = \mathbf{v}^*(\mathbf{x}, t),$$

which minimizes the criterion J[u]. We note that the vectors of phase coordinates x for $t = t_1$ are not given (i.e., the examined problem is one of optimal control with a free right end).

Let V(t, x) be the minimum quality criterion J[u] during translation of system (I) from arbitrary initial point (t, x) (the subscript "0" is omitted) on segment of time $[t, t_1]$, $t \leq t_1$:

$$J^* = J_{\min} = V(t, \mathbf{x}) = \min_{\mathbf{u}} J[\mathbf{u}].$$

In solving the problem by the method of dynamic programming it is best to follow the order of operation given in the summary of general procedures (see Table 3.2). In accordance with Table 3.2 we write the function $H(t, x, \lambda, u)$

---

[11] The symmetrical matrix Q is called positive semidefinite; if all of its eigen values are non-negative or if the quadratic form corresponding to it is non-negative, i.e., $x^T Q x \geqslant 0$ for all $x = (x_1, x_2, \ldots, x_n)^T$. In order that the matrix Q be semi-fixed positive, it is necessary and sufficient that all main (and not just angular!) minors be non-negative:

$$Q\begin{pmatrix} i_1 & i_2 \ldots i_p \\ i_1 & i_2 \ldots i_p \end{pmatrix} > 0 \quad (1 < i_1 < i_2 < \ldots < i_p < n;\ p = 1, 2 \ldots, n).$$

(the Hamiltonian) for the given problem

$$H(t, x, \lambda, u) = f_0(t, x, u) + \lambda^T f(t, x, u) = l_2^T x + l_3^T u + \\ + \frac{1}{2}(x^T Q x + x^T N u + u^T N^T x + u^T P u) + \lambda^T (Ax + Bu + Cf)$$

and we replace conjugate vector λ^T by gradient[12] $V_x(t, x)$ of function $V(t, x)$ for x:

$$H(t, x, V_x, u) = l_2^T x + l_3^T u + \frac{1}{2}(x^T Q x + 2x^T N u + u^T P u) + \\ + V_x(Ax + Bu + Cf).$$

Differential Hamilton-Bellman equation (3.48) in the given case is

$$\frac{\partial V}{\partial t} + \min_u \left\{ l_2^T x + l_3^T u + \frac{1}{2}(x^T Q x + 2x^T N u + u^T P u) + \right. \\ \left. + V_x(Ax + Bu + Cf) \right\} = 0, \qquad \text{(IV)}$$

where the function $V(t, x)$ satisfies boundary condition (3.58'')

$$V(t_1, x) = l_1^T x + \frac{1}{2} x^T R_1 x. \qquad \text{(V)}$$

Since by assumption, $P(t)$ is a positive definite matrix, the minimum $H(t, x, V_x, u)$ for u is achieved at a stationary point, i.e., at the point, where

$$\frac{\partial H}{\partial u} = 0.$$

Hence

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$$u^* = \arg\min_u H(t, x, V_x, u) = -P^{-1}[l_3 + N^T x + B^T V_x^T]. \qquad \text{(VI)}$$

Now by substituting the expression derived for u* into (IV), we find the final form of basic differential equation for dynamic programming (in the given case it will be the differential Hamilton-Jacobi equation, because u* is found from the condition of stability H):

---

[12] Gradient $\partial V(t, x)/\partial x = V_x(t, x)$ of function $V(t, x)$ is considered a row-vector.

$$\frac{\partial V}{\partial t} + V_x Ax - \frac{1}{2} V_x BP^{-1} l_3 - V_x BP^{-1} N^T x - \frac{1}{2} V_x BP^{-1} B^T V_x^T + V_x Cf +$$
$$+ l_2^T x - \frac{1}{2} l_3^T P^{-1} l_3 - l_3^T P^{-1} N^T x - \frac{1}{2} l_3 P^{-1} B^T V_x^T +$$
$$+ \frac{1}{2} x^T Q x - \frac{1}{2} x^T N P^{-1} N^T x = 0. \qquad \text{(VII)}$$

It has been proven that in linear systems with a quadratic quality criterion under the given assumptions relative to matrices Q(t), P(t), N(t), R_1, a solution of equation (VII) with a boundary condition (V) exists and it can be found in the form

$$V(t, x) = \frac{1}{2} x^T R(t) x + q^T(t) x + r(t), \qquad \text{(VIII)}$$

where R(t) is a symmetrical matrix of dimensionality n × n; q(t) is an n-dimensional vector; r(t) is a scalar.

The partial derivatives of the function V(t, x), written in the form (VIII) are

$$\frac{\partial V(t, x)}{\partial t} = \frac{1}{2} x^T \dot{R}(t) x + \dot{q}^T(t) x + \dot{r}(t); \qquad \text{(IX)}$$

$$V_x^T(t, x) = \left(\frac{\partial V(t, x)}{\partial x}\right)^T = R(t) x + q(t); \quad \frac{\partial V(t, x)}{\partial x} = x^T R + q^T. \qquad \text{(X)}$$

By substituting expressions (IX) and (X) into equation (VII) and considering that:

1) for simultaneous multiplication of the arbitrary matrix M on the left and on the right by vector x the relation $x^T Mx = 1/2\ x^T(M + M^T)x$ is valid (i.e., the symmetrical part $1/2(M + M^T)$ of matrix M is separated);

2) the scalar is commutative

whence

$$\frac{1}{2} x^T [\dot{R} + R(A - BP^{-1}N^T) + (A - BP^{-1}N^T)^T R + Q - NP^{-1}N^T -$$
$$- RBP^{-1}B^T R] x + [\dot{q}^T + q^T(A - BP^{-1}N^T) - l_3^T P^{-1} B^T R - q^T BP^{-1} B^T R -$$
$$- l_3^T P^{-1} N^T + l_2^T + (Cf)^T R] x + \dot{r} - \frac{1}{2} q^T BP^{-1} B^T q - l_3^T P^{-1} B^T q + q^T Cf -$$
$$- \frac{1}{2} l_3^T P^{-1} l_3 = 0. \qquad \text{(XI)}$$

Since condition (XI) must be satisfied identically for all x and since the following relation [see (V) and (VIII)] must be satisfied identically for all x when $t = t_1$: /221

$$\frac{1}{2}\mathbf{x}^{\mathrm{T}}R(t_1)\mathbf{x} + \mathbf{q}^{\mathrm{T}}(t_1)\mathbf{x} + r(t_1) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}R_1\mathbf{x} + \mathbf{l}_1^{\mathrm{T}}x,$$

in order to determine matrix R(t), vector $q(t)$ and scalar r(t) we derive the following equations and boundary conditions:

$$1)\ \dot{R} + R(A - BP^{-1}N^{\mathrm{T}}) + (A - BP^{-1}N^{\mathrm{T}})^{\mathrm{T}}R - RBP^{-1}B^{\mathrm{T}}R + Q - NP^{-1}N^{\mathrm{T}} = \dot{R} + RA + A^{\mathrm{T}}R - (RB + N)P^{-1}(N^{\mathrm{T}} + B^{\mathrm{T}}R) + Q = 0; \quad \text{(XII)}$$

$$R(t_1) = R_1. \quad \text{(XII')}$$

$$2)\ \dot{\mathbf{q}}^{\mathrm{T}} + \mathbf{q}^{\mathrm{T}}(A - BP^{-1}N^{\mathrm{T}}) - \mathbf{l}_3^{\mathrm{T}}P^{-1}B^{\mathrm{T}}R - \mathbf{q}^{\mathrm{T}}BP^{-1}B^{\mathrm{T}}R - \mathbf{l}_3^{\mathrm{T}}P^{-1}N^{\mathrm{T}} + \mathbf{l}_2^{\mathrm{T}} + (C\mathbf{f})^{\mathrm{T}}R = 0; \quad \text{(XIII)}$$

$$\mathbf{q}^{\mathrm{T}}(t_1) = \mathbf{l}_1^{\mathrm{T}}. \quad \text{(XIII')}$$

$$3)\ \dot{r} - \frac{1}{2}\mathbf{q}^{\mathrm{T}}BP^{-1}B^{\mathrm{T}}\mathbf{q} - \mathbf{l}_3^{\mathrm{T}}P^{-1}B^{\mathrm{T}}\mathbf{q} + \mathbf{q}^{\mathrm{T}}C\mathbf{f} - \frac{1}{2}\mathbf{l}_3^{\mathrm{T}}P^{-1}\mathbf{l}_3 = 0; \quad \text{(XIV)}$$

$$r(t_1) = 0. \quad \text{(XIV')}$$

The equations derived should be integrated in reverse order from $t = t_1$ to $t = t_0$.

The optimal law of control with feedback has the form

$$\mathbf{u}^*(x, t) = -P^{-1}(t)[(B^{\mathrm{T}}(t)R(t) + N^{\mathrm{T}}(t))\mathbf{x} + B^{\mathrm{T}}(t)\mathbf{q}(t) + \mathbf{l}_3(t)]. \quad \text{(XV)}$$

The solution of certain other problems of optimal control for linear systems with a quadratic quality criterion are given in Table 3.3. In Nos. 1-7 (lines 1-7) of this table, the formulation and solutions of the problem of the synthesis of optimal control law with free boundary conditions on the right end of the trajectory are presented, and in No. 8 the formulation and solution of the problem with assigned boundary conditions on the right end are given. In No.s 1-6, and 8 uniform linear systems are examined; in No. 7 the nonuniform linear system is investigated. No. 1 gives the solution of the problem of synthesis of a nonstable linear system and a nonstable quadratic quality criterion with a fixed end time interval for the control process, and in No. 2 is the solution of a stable (explicitly independent of t) system and a stable quality criterion with a fixed end time interval of the control process. In No. 3 are the solution of a stable system and a stable quality criterion in an unlimited time interval ($[0, \infty]$). No. 4 gives the solution for an unstable system and an unstable quadratic criterion of more general form, than in Nos.

1-3 (the criterion contains type $x^T Nu$ cross terms). No. 5 gives the solution of a problem, in some respect equivalent to the problem in No. 4 (see the fifth column of the table); No. 6 gives the solution for optimization of deviation of a system from a prescribed behavior. No. 7 concerns the case of synthesis of the optimal law of control for a nonuniform linear system, and No. 8 gives the synthesis of the optimal law of control with given boundary conditions on the right end and quadratic criterion of a more general form. Some of the solutions given in Table 3.3 (Nos. 1-4, 6, 7) are special cases of the above-examined problem. /227

Example 3. Synthesis of optimal law of stabilization of aircraft roll by autopilot.

Isolated roll motion of aircraft is described by the following system of linearized differential equations

$$\frac{d\Delta\omega_{x1}}{dt} = b_{11}\Delta\omega_{x1} + b_{18}\Delta\delta_э;$$
$$\frac{d\Delta\gamma}{dt} = \Delta\omega_{x1}. \qquad (I)$$

Here $\Delta\gamma$, $\Delta\omega_{x1}$ -- roll angle and angular velocity of roll in perturbed motion, respectively; $\Delta\delta_a$ is the angle of deflection of the ailerons relative to some programmed position;

$$b_{11} = M^{\omega}_{x1}/J_{x1}; \quad b_{18} = M^{\delta_э}_{x1}/J_{x1},$$

where J_{x1} is the moment of inertia of the aircraft relative to the connected axis Ox_1; M_{x1} is the aerodynamic moment relative to the Ox_1 axis,

$$M^{\omega}_{x1} = \frac{\partial M_{x1}}{\partial\omega_{x1}}, \quad M^{\delta_э}_{x1} = \frac{\partial M_{x1}}{\partial\delta_э}.$$

It is assumed that $M^{\omega}_{x1}x1 < 0$, $M^{\delta}_{x1}a > 0$ and, consequently, $b_{11} < 0$, $b_{18} > 0$.

Let the nominal flight condition be selected so that the coefficients b_{11}, b_{18} can be considered constant. We are required to find the law of control of deviations of the ailerons (the law of roll stabilization)

$$\Delta\delta_э = \Delta\delta_э(\Delta\omega_{x1}, \Delta\gamma),$$

which minimizes the quality criterion

$$J[\Delta\delta_э] = \frac{1}{2}\int_0^{\infty}[q_{11}\Delta\omega^2_{x1} + q_{22}\Delta\gamma^2 + p_1\Delta\delta^2_э]\,dt. \qquad (II)$$

TABLE 3.3.– OPTIMIZATION OF LINEAR SYSTEMS WITH QUADRATIC QUALITY CRITERION IN ABSENCE OF LIMITATIONS OF CONTROL FUNCTIONS

No.	Equation of Motion System	Initial and End Conditions	Quality Criterion $J[u]$	Optimal Law of Control (in terms of Minimum $J[u]$) $u^* = v^*(x, t)$	Optimal Quality Criterion
1	2	3	4	5	6
1	$\dot{x} = A(t)x + B(t)u$ $x = (x_1, \ldots, x_n)^T$ $u = (u_1, \ldots, u_m)^T$ A(t) is matrix of dimensionality n × n. B(t) is matrix of dimensionality n × m.	t_0 - given $x(t_0) = x_0$ - given $t_0 \leqslant t \leqslant t_1$ $x(t_1) = x_1$ - free	$J[u] = \frac{1}{2} x_1^T R_1 x_1 + \frac{1}{2} \int_{t_0}^{t_1} [x^T Q(t) x + u^T P(t) u]\, dt$ R_1, Q(t) - positive semidefinite symmetrical matrices of dimensionality n × n. P(t) - positive definite symmetrical matrix of dimensionality m × m.	$u^* = -P^{-1}(t)B^T(t)R(t)x$, where R(t) is solution of Riccati's Matrix equation: $\dot{R}(t) = -R(t)A(t) - A^T(t)R(t) - Q(t) + R(t)B(t)P^{-1}(t)B^T(t) \times R(t),\ R(t_1) = R_1$ (integration from t_1 to t_0) or $\frac{d}{dt}(R^{-1}(t)) = AR^{-1} + R^{-1}A^T - BP^{-1}B^T + R^{-1}QR^{-1}$ $R^{-1}(t_1) = R_1^{-1}$	$J^* = J_{\min} = V(t_0, x_0) = \frac{1}{2} x^T(t_0) R(t_0) x(t_0)$
2	$\dot{x} = Ax + Bu$ A, B - Constants of matrix of dimensionality n × n and n × m, respectively $x = (x_1, \ldots, x_n)^T$ $u = (u_1, \ldots, u_m)^T$	$t_0 = 0$ $x(t_0) = x_0$ - given $0 \leqslant t \leqslant t_1$ t_1 - given $x(t_1) = x_1$ x_1 - free.	$J[u] = \frac{1}{2} x_1^T R_1 x_1 + \frac{1}{2} \int_0^{t_1} [x^T Q x + u^T P u]\, dt$ Q, R_1 - constant positive semidefinite symmetrical matrices of dimensionality n × n: P - constant positive definite symmetrical matrix of dimensionality m × m.	$u^* = K(t_1 - t)x$, where $K(t_1 - t) = P^{-1} \times B^T R(t_1 - t)$ $R(t_1 - t)$ - solution Riccati's matrix equation: $\frac{dR}{d\tau} = R(\tau)A + A^T R(\tau) + Q - R(\tau)BP^{-1}B^T R(\tau)$ $R(0) = R_1;\ \tau = t_1 - t$ $0 \leqslant \tau \leqslant t_1$	$J^* = J_{\min} = V(t_0, x_0) = \frac{1}{2} x^T(t_0) R(t_1 - t_0) \times x(t_0)$ $t_0 = 0$

TABLE 3.3 (CONTINUED)

No.	Equation of Motion System	Initial and End Conditions	Quality Criterion J[u]	Optimal Law of Control (in terms of Minimum J[u]) $u^* = v^*(x, t)$	Optimal Quality Criterion
1	2	3	4	5	6
3	$\dot{x} = Ax + Bu$ A, B - constants of matrix of dimensionality n × n and n × m, respectively $x = (x_1, \ldots, x_n)^T$ $u = (u_1, \ldots, u_m)^T$	$t_0 = 0$ $x(t_0) = x_0$ - given $0 \leqslant t \leqslant t_1 = \infty$ $x(t_1)$ - free	$J[u] = \frac{1}{2}\int_0^\infty [x^TQx + u^TPu]\,dt$ Q - constant positive definite symmetrical matrix of dimensionality n × n P - constant positive semidefinite symmetrical matrix of dimensionality m × m	$u^* = -Kx$, where $K = P^{-1}B^TR_0$ is constant matrix. R_0 - Steady solution of Riccati's matrix equation, i.e., $R_0 = \lim_{\tau \to \infty} R(\tau)$ where $\frac{dR}{d\tau} = RA + A^TR + Q - RBP^{-1}B^TR;\ R(0) = 0$ R_0 can also be determined from quadratic algebraic Riccati's matrix equation $R_0A + A^TR_0 + Q - R_0BP^{-1}B^TR_0 = 0$ as its only fixed positive solution.	$J^* = J_{min} = V(t_0, x_0) = \frac{1}{2}x_0^TR_0x_0$
4	$\dot{x} = A(t)x + B(t)u$ $A(t), B(t), x, u$ — Matrices and vectors, determined in #1	t_0 - given $x(t_0) = x_0$ - given $t_0 \leqslant t \leqslant t_1$, t_1 - given	$J[u] = \frac{1}{2}x_1^TR_1x_1 + \frac{1}{2} \times$ $\times \int_{t_0}^{t_1} [x^T, u^T] \begin{bmatrix} Q & N \\ N^T & P \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt,$	$u^0 = -P^{-1}[B^TR + N^T]x$, where $\dot{R} = -RA - A^TR + (RB + N)P^{-1}(N^T + B^TR) - Q$ $R(t_1) = R_1$	

TABLE 3.3 (CONTINUED)

No.	Equation of Motion System	Initial and End Equations	Quality Criterion J[u]	Optimal Law of Control (in terms of Minimum J[u]) $u^* = v^*(x, t)$	Optimal Quality Criterion
1	2	3	4	5	6
4		$x(t_1) = x_1$ x_1 - free	where $Q - NP^{-1}N^T \geq 0$ N(t) is matrix of dimensionality n × m P(t) is positive definite matrix of dimensionality m × m R_1 - see #1		
5	$\dot{x} = (A - BP^{-1}N^T)x + Bu$ $A(t)$, $B(t)$, x, u — matrices and vectors, determined in #1 P(t), N(t) - matrices, determined in #4.	t_0 - given x_0 - given $t_0 \leq t \leq t_1$ t_1 - given $x(t_1) = x_1$ x_1 - free	$J[u] = \frac{1}{2} x_1^T R_1 x_1 +$ $+ \frac{1}{2} \int_{t_0}^{t_1} [x^T, u^T] \times$ $\times \begin{bmatrix} Q - NP^{-1}N^T, & 0 \\ 0, & P \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$	$u^* = -P^{-1}B^T Rx -$ $= u^*_{(4)} + P^{-1}N^T x,$ where R and $u^*_{(4)}$ are determined in #4.	

TABLE 3.3 (CONTINUED)

No.	Equation of Motion System	Initial and End Equations	Quality Criterion J[u]	Optimal Law of Control (in terms of Minimum J[u]) $u^* = v^*(x, t)$	Optimal Quality Criterion
1	2	3	4	5	6
6	$\dot{x} = A(t)x + B(t)u$ Matrices and vectors, determined in #1	t_0 - given x_0 - given $t_0 \leqslant t \leqslant t_1$ t_1 - given $x(t_1) = x_1$ x_1 - free	$J[u] = \frac{1}{2}\int_{t_0}^{t_1} [(y(t) - M(t)x)^T Q(t)(y(t) - M(t)x) + u^T P(t) u]\, dt$, where $y(t)$ is given function (desired output signal) $M(t)$ is matrix of dimensionality $n \times n$ $P(t)$, $Q(t)$ - see #2. $M(t)x$ is derived output signal $y = (y_1, y_2, \ldots, y_n)^T$	$u^* = -C(t)x + h(t)$, where $C = P^{-1}B^T R$; $h = -P^{-1}B^T g$, and matrix $R(t)$ and vector $g(t)$ are determined from solutions of equations: $\dot{R} = -RA - A^T R + RBP^{-1}B^T R - M^T Q M$ $R(t_1) = 0$ $\dot{g} = -(A^T - RBP^{-1}B^T) \times g + M^T Q y$ $g(t_1) = 0$	
7	$\dot{x} = A(t)x + B(t)u + f(t)$ $f(t)$ - Known n-dimensional vector, elements $A(t)$, $B(t)$, x, u - Determined in #1.	t_0 - given $x(t_0) = x_0$ -given $t_0 \leqslant t \leqslant t_1$ t_1 - given $x(t_1) = x_1$ - given	$J[u] = \frac{1}{2} x_1^T R_1 x_1 + \frac{1}{2}\int_{t_0}^{t_1} [x^T Q(t) x + u^T P(t) u]\, dt$ R_1, $Q(t)$, $P(t)$ - see #1	$u^* = -P^{-1}B^T(Rx + w)$, where $\dot{R} = -RA - A^T R - Q + RBP^{-1}B^T R$ $R(t_1) = R_1$ $\dot{w} = (RBP^{-1}B^T - A^T)w - Rf$ $w(t_1) = 0$	
8	$\dot{x} = A(t)x + B(t)u$ $A(t)$, $B(t)$, x, u are matrices and vectors, determined in #1.	t_0 - given $x(t_0) = x_0$ - given $t_0 \leqslant t \leqslant t_1$ t_1 - given $Mx(t)_1) = \psi_1$ M - matrix $(q \times n)$ ψ_1 - given q-dimensional vector $q \leqslant n$	$J[u] = \frac{1}{2}\int_{t_0}^{t_1} [x^T Q(t) x + 2x^T N(t) u + u^T P(t) u]\, dt + \frac{1}{2} x_1^T R_1 x_1$ $Q(t)$, $N(t)$, $P(t)$, R_1 — see #4	$u^* = -P^{-1}[N^T + B^T(R - FG^{-1}F^T)]x - P^{-1}B^T FG^{-1}\phi_1$, where $\dot{R} = -RA - A^T R - Q + (RB + N)P^{-1} \times (N^T + B^T R)$ $R(t_1) = R_1$ $\dot{F} = -[A^T - (RB + N) \times P^{-1}B^T]F$ $F(t_1) = M^T$ $\dot{G} = F^T BP^{-1}B^T F$ $G(t_1) = 0$	$J^* = J_{\min} = V(t_0, x_0) = \frac{1}{2} x_0^T (R(t_0) - F(t)G^{-1}(t)F^T \times (t))x_0 + (FG^{-1}\phi_1)^T \times x_0 - \frac{1}{2}\phi_1^T G^{-1}\phi_1$

Here the weight coefficients q_{11}, q_{22} are the "penalties" of the deviations of angular roll velocity and the roll angle, respectively, due to unperturbed motion during infinite interval $[t_0, \infty]$ of time of stabilization; $t_0 = 0$, p_1 is the penalty of "energy" losses to stabilization. It is assumed that q_{11}, q_{22}, p_1 are some positive constants. (In some cases it is advisable to use the following values as their approximate ones:

$$q_{11} = \left[\frac{1}{(\Delta\omega_{x1})_{\max}}\right]^2, \quad q_{22} = \left[\frac{1}{(\Delta\gamma)_{\max}}\right]^2, \quad p_1 = \left[\frac{1}{(\Delta\delta_a)_{\max}}\right]^2,$$

where $(\Delta\omega_{x1})$ max, $(\Delta\gamma)$ max; $(\Delta\delta_a)$ max are the maximum permissible values of the integrated variables in the interval of stabilization).

Let no limitations be imposed on the control variable $\Delta\delta_a$. We shall use data given in Table 3.3 (see row 3) to solve the formulated problem. Since equations (I) in vectorial form are written as /228

$$\begin{bmatrix}\Delta\dot{\omega}_{x1}\\ \Delta\dot{\gamma}\end{bmatrix} = \begin{bmatrix}b_{11} & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta\omega_{x1}\\ \Delta\gamma\end{bmatrix} + \begin{bmatrix}b_{18}\\ 0\end{bmatrix}\delta, \tag{I'}$$

we thus find the symbols of Table 3.3

$$A = \begin{bmatrix}b_{11} & 0\\ 1 & 0\end{bmatrix}, \quad B = \begin{bmatrix}b_{18}\\ 0\end{bmatrix}, \quad Q = \begin{bmatrix}q_{11} & 0\\ 0 & q_{22}\end{bmatrix}, \quad P = p_1.$$

Now the optimal law of stabilization according to Table 3.3 acquires the form

$$\Delta\delta_э = -P^{-1}B^{\mathrm{T}}R_0\begin{bmatrix}\Delta\omega_{x1}\\ \Delta\gamma\end{bmatrix} = -\frac{1}{p_1}[b_{18}, 0]\begin{bmatrix}r_{11} & r_{12}\\ r_{12} & r_{22}\end{bmatrix}\begin{bmatrix}\Delta\omega_{x1}\\ \Delta\gamma\end{bmatrix} = \\ = -(b_{18}r_{11}/p_1)\Delta\omega_{x1} - (b_{18}r_{12}/p_1)\Delta\gamma, \tag{III}$$

where the matrix

$$R_0 = \begin{bmatrix}r_{11} & r_{12}\\ r_{12} & r_{22}\end{bmatrix}$$

is the only positive definite solution of Riccati's algebraic quadratic matrix equation

$$R_0A + A^{\mathrm{T}}R_0 + Q - R_0BP^{-1}B^{\mathrm{T}}R_0 = 0 \tag{IV}$$

or

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}\begin{bmatrix} b_{11} & 0 \\ 1 & 0 \end{bmatrix}+\begin{bmatrix} b_{11} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}+\begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}-$$
$$-\frac{1}{p_1}\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}\begin{bmatrix} b_{18} \\ 0 \end{bmatrix}[b_{18}, 0]\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}=\begin{bmatrix} r_{11}b_{11}+r_{12}, & 0 \\ r_{12}b_{11}+r_{22}, & 0 \end{bmatrix}+ \qquad \text{(V)}$$
$$+\begin{bmatrix} r_{11}b_{11}+r_{12} & r_{12}b_{11}+r_{22} \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}-\frac{1}{p_1}\begin{bmatrix} b_{18}^2r_{11}^2 & b_{18}^2r_{11}r_{12} \\ b_{18}^2r_{11}r_{12} & b_{18}^2r_{12}^2 \end{bmatrix}=0.$$

This matrix equation is equivalent to the system of three linear algebraic equations relative to r_{11}, r_{12}, r_{22}:

$$\left.\begin{aligned} 2(r_{11}b_{11}+r_{12})+q_{11}-\frac{1}{p_1}b_{18}^2r_{11}^2=0, \\ r_{12}b_{11}+r_{22}-\frac{1}{p_1}b_{18}^2r_{11}r_{12}=0; \\ q_{22}-\frac{1}{p_1}b_{18}^2r_{12}^2=0. \end{aligned}\right\} \qquad \text{(VI)}$$

The solutions of system (VI) are:

$$\left.\begin{aligned} r_{12}=\pm\sqrt{p_1q_{22}/b_{18}^2}, \\ r_{11}=b_{11}p_1/b_{18}^2\pm\sqrt{(b_{11}p_1/b_{18}^2)^2+2p_1r_{12}/b_{18}^2+q_{11}p_1/b_{18}^2}, \\ r_{22}=(b_{18}^2/p_1)r_{11}r_{12}-b_{11}r_{12}= \\ =\pm(b_{18}^2/p_1)\sqrt{(b_{11}p_1/b_{18}^2)^2+2p_1r_{12}/b_{18}^2+q_{11}p_1/b_{18}^2}r_{12}. \end{aligned}\right\} \qquad \text{(VII)}$$

Since only the positive definite matrix R_0 corresponds to the optimal solution its elements must satisfy the Sylvester criterion (see Section 3.4.4), i.e., /229

$$r_{11}>0, \quad r_{22}>0, \quad r_{11}r_{22}-r_{12}^2>0.$$

Since $b_{11} < 0$, it is clear that in the second equation (VII) the "plus" sign should be written in front of the radical. In addition, it follows from the third equation (VII) that $r_{12} > 0$. Thus, finally

$$\left.\begin{aligned} r_{12}=\sqrt{p_1q_{22}/b_{18}^2}; \\ r_{11}=b_{11}p_1/b_{18}^2+\sqrt{(b_{11}p_1/b_{18}^2)^2+2(p_1/b_{18}^2)\sqrt{p_1q_{22}/b_{18}^2}+q_{11}p_1/b_{18}^2}; \\ r_{22}=(b_{18}^2/p_1)\left[(b_{11}p_1/b_{18}^2)^2+2(p_1/b_{18}^2)\sqrt{p_1q_{22}/b_{18}^2}+\right. \\ \left.+q_{11}p_1/b_{18}^2\right]^{1/2}\sqrt{p_1q_{22}/b_{18}^2}. \end{aligned}\right\} \qquad \text{(VIII)}$$

Upon considering (VIII) the law of optimal stabilization becomes

$$\begin{aligned}\Delta\delta_э=&-(b_{18}/p_1)\left\{b_{11}p_1/b_{18}^2+\left[\left(b_{11}p_1/b_{18}^2\right)+2\left(p_1/b_{18}^2\right)\times\right.\right.\\&\times\sqrt{p_1q_{22}/b_{18}^2}+q_{11}p_1/b_{18}^2\Big]^{1/2}\Big\}\Delta\omega_{x1}-(b_{18}/p_1)\times\\&\times\sqrt{p_1q_{22}/b_{18}^2}\,\Delta\gamma=-\left\{b_{11}/b_{18}+\left[(b_{11}/b_{18})^2+2\times\right.\right.\\&\left.\left.\times\sqrt{q_{22}/(p_1b_{18}^2)}+q_{11}/p_1\right]^{1/2}\right\}\Delta\omega_{x1}-\sqrt{q_{22}/p_1}\Delta\gamma=\\&=-k_\omega\Delta\omega_{x1}-k_\gamma\Delta\gamma,\end{aligned}\tag{IX}$$

where

$$\begin{aligned}k_\omega=&\left\{M_{x1}^{\omega_{x1}}/M_{x1}^{\delta_э}+\left[\left(M_{x1}^{\omega_{x1}}/M_{x1}^{\delta_э}\right)^2+2\sqrt{\left(J_{x1}/M_{x1}^{\delta_э}\right)^2q_{22}/p_1}+q_{11}/p_1\right]^{1/2}\right\},\\k_\gamma=&\sqrt{q_{22}/p_1}-\end{aligned}\tag{X}$$

are transfer numbers (coefficients in the feedback circuit) of the autopilot.

The minimal quality criterion (see Table 3.3) is determined by the expression

$$\begin{aligned}J_{\min}=V(\Delta\omega_{x1_0},\ \Delta\gamma_0)=&\frac{1}{2}\left[\Delta\omega_{x1_0},\ \Delta\gamma_0\right]\begin{bmatrix}r_{11}r_{12}\\r_{12}r_{22}\end{bmatrix}\begin{bmatrix}\Delta\omega_{x1_0}\\\Delta\gamma_0\end{bmatrix}=\\=&\frac{1}{2}\left[r_{11}\Delta\omega_{x1_0}^2+2r_{12}\Delta\omega_{x1_0}\Delta\gamma_0+r_{22}\Delta\gamma_0^2\right]=\\=&\frac{p_1}{2}\left(J_{x1}/M_{x1}^{\delta_э}\right)\left[k_\omega\Delta\omega_{x1_0}^2+2k_\gamma\Delta\omega_{x1_0}\Delta\gamma_0+\right.\\&\left.+k_\gamma\left(J_{x1}/M_{x1}^{\delta_э}\right)\left(k_\omega-M_{x1}^{\omega_{x1}}/M_{x1}^{\delta_э}\right)\Delta\gamma_0^2\right].\end{aligned}\tag{XI}$$

3.6. Necessary Conditions of Optimality of Singular Control

3.6.1. Brief Formulation of Problem

In solving problems of flight mechanics one encounters cases when control u enters linearly in differential equations of the mathematical model

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(t,\ \mathbf{x},\ \mathbf{u})=\boldsymbol{\gamma}(\mathbf{x},\ t)+R(\mathbf{x},\ t)\,\mathbf{u},\tag{3.59}$$

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where

$$\begin{aligned}&\mathbf{x}=(x_1,\ x_2,\ \ldots,\ x_n)^T,\ \mathbf{x}\in X^n;\\&\mathbf{u}=(u_1,\ u_2,\ \ldots,\ u_m)^T,\ \mathbf{u}\in U^m;\\&\boldsymbol{\gamma}=(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n)^T;\\&R=\{r_{ij}(t,\ \mathbf{x})\}\quad(i=1,\ 2,\ \ldots,\ n,\quad j=1,\ 2,\ \ldots,\ m);\\&t\in[t_0,\ t_1],\end{aligned}$$

but the quality criterion is

$$J[\mathbf{u}, t_0, t_1, \mathbf{x}_0, \mathbf{x}_1] = \Phi(t_0, t_1, \mathbf{x}_0, \mathbf{x}_1) + \\ + \int_{t_0}^{t_1} f_0(t, \mathbf{x}, \mathbf{u})\,dt = \Phi(t_0, t_1, \mathbf{x}_0, \mathbf{x}_1) + \\ + \int_{t_0}^{t_1} [\gamma_0(\mathbf{x}, t) + \mathbf{u}^{\mathrm{T}}\mathbf{r}_0(\mathbf{x}, t)]\,dt, \tag{3.60}$$

where

$$\mathbf{r}_0 = (r_{01}, r_{02}, \ldots, r_{0m})^{\mathrm{T}};$$

$$\mathbf{u}^{\mathrm{T}}\mathbf{r}_0 = \sum_{j=1}^{m} r_{0j}u_j.$$

The Hamilton function H for (3.59), (3.60) is

$$H = \sum_{i=0}^{n} \lambda_i f_i = \sum_{i=0}^{n} \lambda_i \gamma_i(\mathbf{x}, t) + \sum_{i=0}^{n} \lambda_i \times \\ \times \sum_{j=1}^{m} r_{ij}u_j = \sum_{i=0}^{n} \lambda_i \gamma_i(\mathbf{x}, t) + \sum_{j=1}^{m} \left(\sum_{i=0}^{n} \lambda_i r_{ij} \right) u_j. \tag{3.61}$$

If U^m is an m-dimensional rectangle:

$$U^m = \{\mathbf{u} = (u_1, u_2, \ldots, u_m)^{\mathrm{T}} \mid a_1 \leqslant u_1 \leqslant b_1,\ a_2 \leqslant u_2 \leqslant b_2, \\ \ldots,\ a_m \leqslant u_m \leqslant b_m\},\quad a_j < b_j \quad (j = 1, 2, \ldots, m)$$

(a_j, b_j may depend on t) because of the maximum principle (Section 3.4.3) optimal control is determined from the condition

$$\mathbf{u} = \arg\min_{\mathbf{u} \in U^m} H(t, \mathbf{x}, \mathbf{u}, \lambda) \tag{3.62}$$

for minimization of $J[\mathbf{u}]$ or

$$u_j = \begin{cases} a_j \text{ when } \sum_{i=0}^{n} \lambda_i r_{ij} > 0; \\ b_j \text{ when } \sum_{i=0}^{n} \lambda_i r_{ij} < 0. \end{cases} \tag{3.63}$$

For certain values of x and λ the function H in (3.61) may be explicitly independent of any component of u_j or segment $[\tau_1, \tau_2]$, $\tau_2 - \tau_1 > 0$. In this case the following relation is satisfied (see also Figure 3.9) /231

$$\Phi_j(\lambda, \mathbf{x}, t) = \sum_{i=0}^{n} \lambda_i r_{ij}(\mathbf{x}, t) \equiv 0, \tag{3.64}$$

which formally coincides with the condition:

$$\frac{\partial H}{\partial u_j} = \sum_{i=0}^{n} \lambda_i r_{ij}(x, t) \equiv 0 \tag{3.65}$$

on segment $[\tau_1, \tau_2]$.

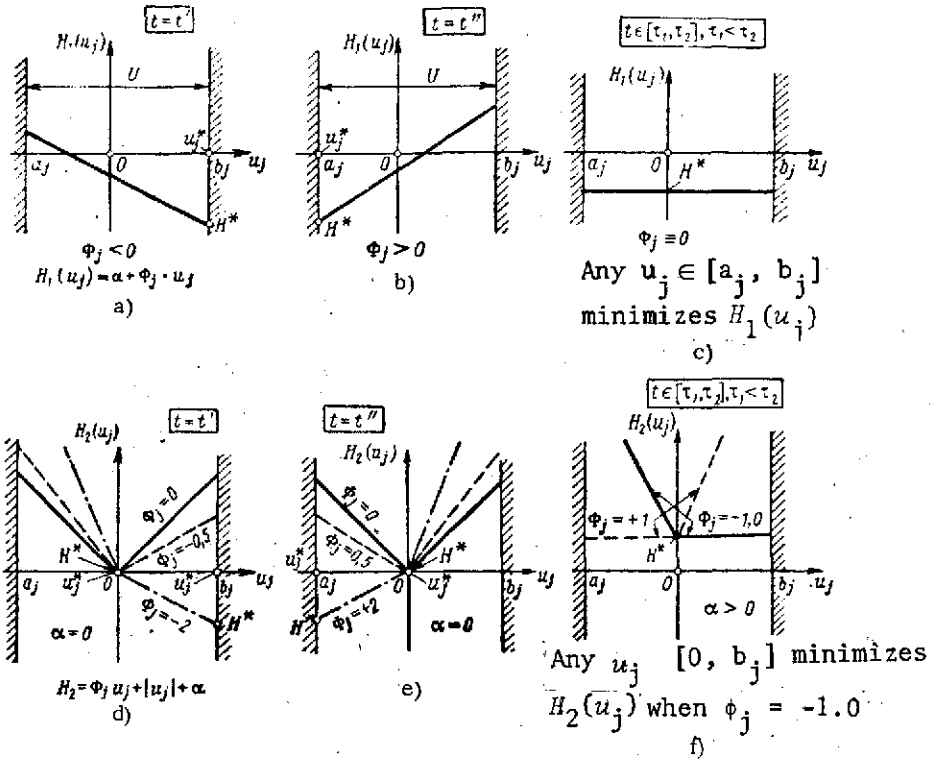

Figure 3.9. Behavior of the Hamiltonians $H_1(u_j) = \alpha + \Phi_j u_j$ and $H_2(u_j) = \Phi_j u_j + |u_j| + \alpha$ in relation to Φ_j:

a, b, d, e, Rigorous minimum (regular control);
c, f, non-rigorous minimum (singular control).

The segment $[\tau_1, \tau_2]$, in which relation (3.64) is valid, is called a segment of singular control for the component u_j, and optimal control $u_j^*(t)$ in such segment (if it exists) is called a singular optimal control. This term is used /232 because of the fact that Hamiltonian H does not depend on u_j, the optimal control cannot be found directly using the maximum principle. In addition, when condition (3.64) is satisfied neither the necessary conditions of the classical calculus of variations (see Section 3.9.2), nor the necessary conditions of

dynamic programming (see Section 3.5.2) can be used for direct calculation of the component u_j* although all of these conditions formally are satisfied.

Thus for example, if Hamiltonian H does not depend on control u_j then H reaches a maximum for all of u_j.

Conditions (3.64) cannot establish the contrast between controls u_j, which minimize or maximize to functional J[u]. The relation the

$$\det\left\{\frac{\partial^2 H}{\partial u_i \partial u_j}\right\} \equiv 0 \quad (i, j = 1, 2, \ldots, m), \text{ на } [\tau_1, \tau_2], \tag{3.66}$$

is satisfied on the segment of singular control, which indicates that the Hilbert condition of nondegeneracy of a variational problem (see Section 3.9.4) is violated. Problems in which condition (3.66) is valid, are called degenerate in the classical calculus of variations. If set U^m is closed and bounded, two conditions of optimal control can be encountered in problems of degeneracy: regular, when u is determined from the maximum principle (as, for example, (3.63)), and singular, when u cannot be found from the maximum principle [as for example, when (3.64) is satisfied] and when a special procedure is required for its determination.

3.6.2. Procedure for Finding Singular Control

The general theory of degenerate variation problems is still insufficiently developed. The case of singular control by a single component u_j is most thoroughly investigated. In this case the solution can be derived in the following manner.

Condition (3.65) shows that the condition of singular control on segment $[\tau_1, \tau_2]$ (the segment of singular control) is valid if

$$\frac{\partial H}{\partial u_j} \equiv \sum_{i=0}^{n} \lambda_i r_{ij}(t, x) \equiv 0.$$

Successive differentiation of this relation in terms of t leads to the relations

$$\frac{d^k}{dt^k}\left(\frac{\partial H}{\partial u_j}\right) \equiv 0 \text{ on } [\tau_1, \tau_2] \quad (k = 0, 1, 2, \ldots). \tag{3.67}$$

It can be shown that the first non-zero value of

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$$\frac{\partial}{\partial u_j}\left[\frac{d^k}{dt^k}\left(\frac{\partial H}{\partial u_j}\right)\right]$$

is possible only when k is even. We shall denote it as $k = k_{min} = 2p$. The number p is called the order of degeneracy (singularity) of the variational problem (optimal control).

When $k = 2p$, control u_j enters in $d^k/dt^k(\partial H/\partial u_j)$ explicitly. Now the singular optimal control u_j^* can be found from the condition

$$\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u_j}\right)=0 \quad \text{on } [\tau_1, \tau_2], \tag{3.68}$$

which is linear for u_j (in view of the linearity of system (3.59) in terms of u). The equations of the conjugate system (see Section 3.4.4) in the given case are

$$\frac{d\lambda_s}{dt}=-\sum_{i=0}^{n}\lambda_i\frac{\partial\gamma_{it}}{\partial x_s}+\sum_{j=1}^{m}\left(\sum_{i=0}^{n}\lambda_i\frac{\partial r_{ij}}{\partial x_s}\right)u_j. \tag{3.69}$$

Upon considering that all the remaining components of vector u are regular, i.e., are determined by relation of the type (3.63), condition (3.68) can be written as

$$\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u_j}\right)=M_1(\mathrm{x},\ \lambda,\ t)+u_jM_2(\mathrm{x},\ \lambda,\ t)=0, \tag{3.70}$$

from which the singular control of component u_j can also be found:

$$u_j=-\frac{M_1(\mathrm{x},\ \lambda,\ t)}{M_2(\mathrm{x},\ \lambda,\ t)}.$$

3.6.3. The Necessary Condition of Optimality of Singular Control

For the minimum quality criterion J[u] on singular control u_j^* in problem (3.59)-(3.60) the following necessary condition must be satisfied

$$(-1)^p\frac{\partial}{\partial u_j}\left[\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u_j}\right)\right]\geqslant 0, \quad p=0,\ 1,\ 2,\ \ldots \tag{3.71}$$

When maximizing the quality criterion the sign in inequality (3.71) should be reversed.

We note that when p = 0, i.e., for nondegenerate problems, this condition converts to the condition $\partial^2H/\partial u_j^2 \geqslant 0$ (when m = 1) and, thus, (3.71) is the

analog of the Legendre-Clebsch condition (see Section 3.9.4) for singular (degenerate) extrema (for unidimensional control u_j). When p = 1, condition (3.71) is /234

$$\frac{\partial}{\partial u_j}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u_j}\right)\right] \leqslant 0. \tag{3.71'}$$

3.6.4. Necessary Conditions at Conjugation Points of Singular and Regular Controls

The results of Sections 3.6.2 and 3.6.3 are applicable if the optimal singular controls $u_j^*(t)$ are interior points of set U^m on segment $[\tau_1, \tau_2]$. The necessary conditions for conversion from regular optimal control to singular optimal control in the case when U^m is an m-dimensional rectangle $a_j(t) \leqslant u_j(t) \leqslant b_j(t)$, and τ_1 is the initial moment of the conversion, determined by the following inequalities:

$$[M_1(x, \lambda, t) + b_j(t) M_2(x, \lambda, t)]_{\tau_1} < 0 \tag{3.72}$$

(the necessary condition for the possibility of conversion of regular control from the upper bound $u_j(t) = b_j(t)$ to singular optimal control) and

$$[M_1(x, \lambda, t) + a_j(t) M_2(x, \lambda, t)]_{\tau_1} > 0 \tag{3.73}$$

(the necessary condition for the possibility of conversion of regular control from the lower bound $u_j(t) = a_j(t)$ to singular optimal control).

The requirement of combined satisfaction of conditions (3.72) and (3.73) can be represented as the inequality

$$\frac{\partial}{\partial u_j}\left[\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u_j}\right)\right]_{\tau_1} \leqslant 0. \tag{3.74}$$

This condition is necessary for the possibility of conversion from both boundaries of regular control to singular control. Necessary condition (3.74) is easier to check because it does not involve the calculation $M_1(x, \lambda, t)$. However, it should be recalled that it is weaker than conditions (3.72) and (3.73) because the latter are not derived from it.

3.7. Necessary Conditions of Control Optimality in Problems with Inequality Type Limitations, Which Contain Only Phase Coordinates x

In flight mechanics many problems are encountered in formulation of an optimal trajectory, when it is necessary to consider limitations on the range of the permissible values of phase coordinates. For example, when analyzing the climb of an airplane or launch trajectories, constraint is often placed on velocity head q: /235

$$q=\frac{\varrho(h(t))v^2(t)}{2}\leqslant q_{\text{kn}},$$

i.e.,

$$q(h(t),\ v(t),\ t)-q_{\text{kn}}\leqslant 0.$$

Limitations on the permissible flight altitude h and mass m of aircraft are also typical of aircraft motion

$$h(t)\geqslant 0;\ m(t)\geqslant \ldots$$

In the general case limitations of this type can be written as

$$\psi(t,\ \mathbf{x})\geqslant 0. \tag{3.75}$$

where

$$\psi=(\psi_1,\ \psi_2,\ \ldots,\ \psi_{\mu_1})^T;\quad \mathbf{x}=(x_1,\ x_2,\ \ldots,\ x_n)^T.$$

3.7.1. Brief Formulation of the Problem

Let the evolution of the examined system S be described by the vectorial differential equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(t,\ \mathbf{x},\ \mathbf{u}), \tag{3.76}$$

where

$$\mathbf{f}=(f_1,\ f_2,\ \ldots,\ f_n)^T;\quad \mathbf{x}=(x_1,\ x_2,\ \ldots,\ x_n)^T;\quad \mathbf{u}=(u_1,\ u_2,\ \ldots,\ u_m)^T;$$
$$\mathbf{u}\in U^m.$$

U^m is some closed and bounded region in space R^m.

Given: initial value

$$\mathbf{x}(t_0)=\mathbf{x}_0, \tag{3.77}$$

time interval $[t_0,\ t_1]$,

quality criterion

$$J[\mathbf{u}]=\Phi(t_1,\ \mathbf{x}(t_1))+\int_{t_0}^{t_1} f_0(t,\ \mathbf{x},\ \mathbf{u})dt. \tag{3.78}$$

It is required to find piece-wise continuous control $u(t)\in U^m$, which translates the initial state $(t_0,\ x_0)$ to some end point $(t_1,\ x(t_1))$, which satisfies conditions

$$\mathbf{q}(t_1,\ \mathbf{x}(t_1))=0,\quad \mathbf{q}=(q_1,\ q_2,\ \ldots,\ q_l)^{\mathrm{T}},\quad l<n+1 \tag{3.79}$$

and minimizes functional J[u] on trajectories which satisfy the conditions

$$\psi(t,\ \mathbf{x})\geqslant 0,\quad \psi=(\psi_1,\ \psi_2,\ \ldots,\ \psi_{\mu_1})^{\mathrm{T}}. \tag{3.75'}$$

Here the values of the function ψ_i do not depend explicitly on control u. /236
It is assumed that f, f_0, ψ have continuous derivatives as high as the second order.

3.7.2. Necessary Conditions of Optimality

In the statement of Section 3.7.1 the entire optimal trajectory of the flight in the general case can consist of two types of segments: those which lie entirely within the permissible region, and those which lie on the boundary of the permissible region (Figure 3.10). The number of such sections and their alternation depends on the specific problem and the boundary conditions. On segments located entirely within the permissible region conditions (3.75) are satisfied as rigorous inequalities

$$\psi(t,\ \mathbf{x})>0.$$

For these segments the maximum principle formulated in Section 3.4.3 is valid.

On segments which lie on the boundary of the permissible region, one or several conditions of the type (3.75) are satisfied as equalities. These segments are called boundary segments, and for them the maximum principle of Section 3.4.3 is not valid. The given problem differs from problems of Section 3.4.1 by the presence of these segments.

Several equivalent approaches to derivation of the necessary conditions of optimization for segments on the boundary $\psi(t, x) = 0$ are known. Being equivalent, these approaches lead to different calculation procedures for obtaining a solution.

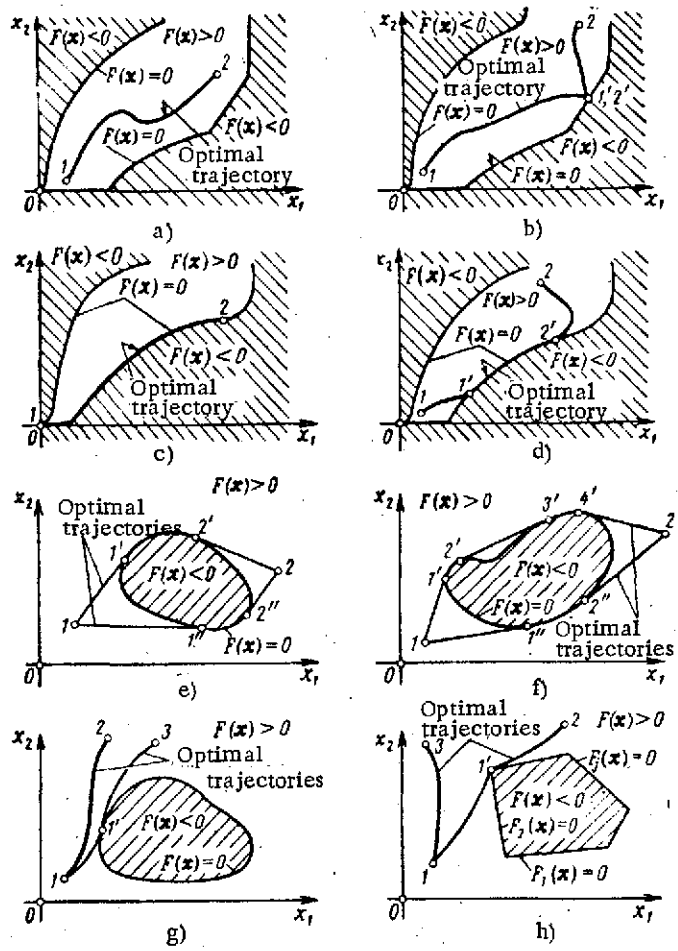

Figure 3.10. Types of Possible Optimal Trajectories in Problems With Constrained Phase Coordinates.
a,b,c,d, Cases where the permissible trajectories are located within a specific region (not necessarily closed); a, Trajectory which lies entirely within the permissible regions; b, Trajectory having a point in common with the boundary of the region (a type of reflection from the boundary); c, Trajectory which lies entirely on the boundary; d, Trajectory which is located partially on the boundary; e,f,g,h, Cases where the permissible trajectories are located outside of specific region; e, Case of two trajectories which achieve the relative minimum in problem of shortest path on plane; f, Case of nonconvex prohibited region; trajectories with several ascending and descending segments; g, 1-2 trajectory with no points in common with boundary; 1-3, Trajectory with one point (contact) in common with boundary; h, Case of nonsmooth boundary of permissible region; 1, Initial point of trajectory; 2, End point of trajectory; 1', Point of arrival on boundary; 2', The Point of departure from boundary.

We shall examine a case of one scalar limitation of the form

$$\psi_i(t, x) \geqslant 0.$$

First Type of Necessary Optimality Conditions for Boundary Segments of Trajectory

For the sake of simplicity a case in which only one of the limitations of type (3.75), satisfied as an inequality, is examined here (for example, limitation of ψ_1). Let this limitation be

$$\psi_1(t, x)=0 \tag{3.80}$$

such that the complete derivative of time

$$\frac{d\psi_1(t, x)}{dt}=\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_1}{\partial x}\dot{x}=\frac{\partial\psi_1}{\partial t}+\left(\frac{\partial\psi_1}{\partial x}\right)f(t, x, u) \tag{3.81}$$

explicitly contains control u.

The necessary and sufficient condition of the fact that (3.80) is valid on some non-zero segment $[t_1', t_2']$, reduces to the equation /238

$$\dot{\psi}_1=\frac{d\psi_1(t, x)}{dt}=\frac{\partial\psi_1}{\partial t}+\left(\frac{\partial\psi_1}{\partial x}\right)f(t, x, u)=\dot{\psi}_1(t, x, u)=0. \tag{3.82}$$

The Hamiltonian H_1 is written for the boundary sections

$$H_1=H+\beta\dot{\psi}_1(t, x, u), \tag{3.83}$$

where $H=\lambda_0 f_0+\sum_{i=1}^{n} f_i\lambda_i$;

$\beta = 0$ on segments where $\psi_1 > 0$;

$\beta \neq 0$ on segments where $\psi_1 = 0$.

Now the necessary conditions for the boundary segment coincide with the necessary conditions of Section 3.8.3 with replacement in conditions (3.98), (3.100), (3.104) of function $\varkappa$ by $\dot{\psi}_1$. The contrast between this problem and the one in Section 3.8.2, consists in conditions which are imposed on the variables at the points of arrival of the trajectory on the boundary and departure from it. At these points the conjugate variable $\lambda_i(t)$ can experience discontinuities. If there are two segments, then all the conjugate variables are continuous. In this case the condition $\psi_1(t, x) = 0$ can be interpreted either as a constraint imposed on the initial conditions (t_0, x_0) or as a

constraint imposed on the end condition (t_1, x_1), depending on the order of sequence of segments with $\psi_1 > 0$ and $\psi_1 = 0$.

When there are three segments, if a boundary segment goes first, then the segment with $\psi_1 > 0$ and then a boundary section again, then the factors are also continuous on the entire trajectory. With all other sequences of segments, when there are more than three in the sequence, the conjugate variables have a jump type discontinuity. This jump of $\lambda_i(t)$ may occur at either end of the boundary segment, and here the factors can be made continuous at the other end (the choice of the end, at which the jump occurs, is trivial). If this end is selected at moment of time t'_2, the conditions of the jump are

$$\lambda^+(t_2)=\lambda^-(t_2)-C\frac{\partial\psi_1(t'_2)}{\partial x}; \tag{3.84}$$

$$H^+(t_2')=H^{-1}(t_2')+C\frac{\partial\psi_1(t_2')}{\partial t}; \tag{3.85}$$

$$\psi_1^-(t_2')=0, \tag{3.86}$$

where C is an arbitrary constant; the indices "+" and "-" denote the limits on the right and left sides, respectively.

If conditions (3.84) are substituted into (3.85) the coefficient for C is /239
ψ_1 and, therefore, condition (3.85) does not depend on C, but contains only the values $\lambda - (t'_2)$. Following this substitution equation (3.85) can be used as an equivalent necessary condition.

In the given problem the solution of $x(t)$, $\lambda(t)$ depends on λ_{i0}, and C depends on the parameters:

$$x=x(t, \lambda_{i0}, C); \quad \lambda=\lambda(t, \lambda_{i0}, C).$$

At each point of discontinuity a new constant C must be added to the continuity of the conjugate variables. The magnitude of C cannot be determined in advance from the necessary conditions and is an additional parameter which determines the point of departure. Since the number of boundary segments is not known a priori the problem becomes one with a variable number of parameters, which substantially complicates the practical solution, even with the computer.

Example 5. Let there be three segments of an optimal trajectory, which follow the order:

Segment 1 - Trajectory in open space, $\psi_1 > 0$;

Segment 2 - Boundary trajectory, $\psi_1 = 0$;

Segment 3 - Again a trajectory in open space, $\psi_1 > 0$.

The necessary conditions at the end point give (n + 1) equations relative to (n + 2) unknowns for λ_{i0}, t_1, C. Conditions (3.85), (3.86) and

$$\beta(t_2' + 0) = 0 \tag{3.87}$$

determine point t_2' and give an additional equation relative to the unknowns λ_{i0}, t_1, C. Thus the problem amounts to finding the solution of (n + 2) equations with (n + 2) unknowns.

If there are more than three segments the problem reduces to a multiple-point boundary problem.

Second Type of Necessary Conditions of Control Optimality on Boundary Segments

Let t_{ar} be the time of arrival of the trajectory on the boundary of the permissible region, t_{de} be the time of departure from this boundary. The Hamiltonian H_2 for the boundary segments can be presented in the following form:

$$H_2 = \lambda_0 f_0 + \sum_{i=1}^{n} \lambda_i f_i + \beta_1 \psi_1 + \beta_2 \dot{\psi}_1 = H + \beta_1 \psi_1 + \beta_2 \dot{\psi}_1,$$

where

$$\beta_1 = \beta_2 = 0, \quad \text{if} \quad \psi_1 > 0;$$
$$\beta_1 \neq 0, \ \beta_2 \neq 0, \quad \text{if} \quad \psi_1 = 0,$$

and $\dot{\psi}_1$ is determined by the right side of relation (3.81).

In the boundary segment (i.e., when $t_{ar} \leq t \leq t_{de}$ on the optimal trajectory, the conditions /240

$$\dot{x} = \left(\frac{\partial H_2}{\partial \lambda}\right)^T,$$
$$\dot{\lambda} = -\left(\frac{\partial H_2}{\partial x}\right)^T, \tag{3.88}$$
$$\psi_1 = 0, \quad \dot{\psi}_1 = 0.$$

are satisfied.

The optimal control on the boundary segment is determined from the condition of minimum H for $u \in U_1^m(t, x)$, where $U_1^m(t, x)$ is that part of u from region U^m, which satisfies the condition $\dot{\psi}_1(t, x, u) = 0$.

If the minimum H for u in region $U_1^m(t, x)$ is reached at an interior point, then

$$\frac{\partial H_2}{\partial u} = \frac{\partial H}{\partial u} + \beta_2 \frac{\partial}{\partial u}(\dot{\psi}(t, x, u)) = 0, \quad \psi_2(t, x) = 0, \quad \dot{\psi}_1(t, x, u) = 0.$$

The values of the vector λ and the Hamiltonian H_2 are continuous at the arrival point on the boundary of the permissible region:

$$\lambda(t_{in} + 0) = \lambda(t_{in} - 0); \quad H_2(t_{in} + 0) = H_2(t_{in} - 0).$$

The remaining, insufficient boundary conditions can be found from the general conditions of transversality in Section 3.4.3. In particular, it follows from these conditions that when $t = t_1$

$$\lambda(t_1) = \left(\frac{\partial L}{\partial x}\right)^T\Big|_{t=t_1}; \quad L = \Phi(t_1, x(t_1)) + \mu^T q(t_1, x(t_1));$$

$$\frac{\partial L}{\partial t_1} + H_2(t_1) = 0 \qquad \text{(if } t_1 \text{ is not given)}.$$

In addition the given boundary condition (3.79) must be added to these conditions:

$$q(t_1, x(t_1)) = 0.$$

3.8. Necessary Conditions of Control Optimality in Problems with Inequality Type Limitations Which Contain Simultaneously Phase Coordinates x and Control u

In flight mechanics problems in which the permissible values of the control functions should not exceed limits which depend on the current condition of the system, are frequently encountered.

The limitations of the examined type can be written in the form

$$\varkappa(t, x, u) \leqslant 0, \tag{3.89}$$

where κ explicitly depends on the states of x and control u. The maximum principle formulated in Section 3.4.3 is valid only for inequalities of the the type /241

$$\varkappa_i(t, u) \leqslant 0, \tag{3.90}$$

i.e., which explicitly do not contain phase coordinates x.

Presented below is a formulation of the maximum principle which is suitable for limitations of the type (3.89).

3.8.1. Brief Formulation of Problem

Let the evolution of system S be described by the vectorial differential equation

$$\frac{dx}{dt} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \tag{3.91}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is an n-dimensional vector of state

$\mathbf{u} = (u_1, u_2, \ldots, u_m)^T$ is an m-dimensional control vector.

The following limitations are imposed on the control vector u:

$$\varkappa(t, \mathbf{x}, \mathbf{u}) \geqslant 0, \tag{3.92}$$

where $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\nu_1})^T$ is a ν_1-dimensional vector and the number of relations which are simultaneously satisfied as equalities does not exceed m.

In region U^m of permissible values u depends on t, x: $U^m = U^m(t, x)$ and is given by equation (3.92). It is assumed that vector u explicitly enters in equation (3.92).

At the initial moment of time $t = t_0$ the state of the system is given:

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{3.93}$$

It is required to change system S from state x_0 to some terminal state, determined by the relations

$$\mathbf{q}(t_1, \mathbf{x}(t_1)) = 0. \tag{3.94}$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_{l_2}), \quad l_2 \leqslant n+1.$

It is required to find the permissible piece-wise continuous vector u(t) which satisfies (3.92), such that the functional

$$J[\mathbf{u}] = \Phi(t_1, \mathbf{x}(t_1)) + \int_{t_0}^{t_1} f_0(t, \mathbf{x}, \mathbf{u})\,dt \tag{3.95}$$

acquires the minimal value in solutions of system (3.91).

Comment. The solutions x(t) of system (3.91) are assumed to be continuous and to have at least absolutely continuous derivatives. The points t_α where one or more components of the vector u experience a discontinuity of the first kind are called cusps. The point t_s, where the symbol ">" changes to "=" (or conversely) in one or several limitations (3.92), are called the connecting points.

3.8.2. Types of Boundary Conditions

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The problem in which $\Phi(t_1, x(t_1)) \equiv 0$, and the boundary conditions (3.94) are

$$x_l(t_1) - x_{l1} = 0 \quad (l = 1, 2, \ldots, l_2 \leqslant n), \tag{3.96}$$

or

$$x_l(t_1) - x_{l1} = 0 \quad (l = 1, 2, \ldots, l_2 - 1 \leqslant n),$$
$$t_1 - t_{kn} = 0, \tag{3.97}$$

where x_{il}, t_{kn} are known numbers, is sometimes called simplest. When $l_2 = n$, conditions (3.96) lead to a problem with a fixed right end and free time. When $l_2 < n$ conditions (3.96) lead to a problem with partially free right end and free time t_1. Conditions of type (3.97) pertain to the problem with fixed time $t_1 = t_{kn}$ and a partially free right end of the trajectory.

3.8.3. Necessary Conditions of Optimality

If $u^*(t) \in U^m(x, t)$ [U^m is determined by conditions (3.92)] is the control, which minimizes the functional J[u], such constants $\lambda_0 = 1$, $\mu = (\mu_1, \ldots, \mu_{l2})^T$ will be found that are not all equal to zero, and such constant vectors $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))^T$ which do not simultaneously vanish (continuous for $[t_0, t_1]$) and $\beta(t) = (\beta_1(t), \ldots, \beta_{\nu_1}(t))^T$ (continuous for $[t_0, t_1]$ everywhere with the exception, perhaps, of points of discontinuity of control u(t), where, however, there exist individual right and left limits), so that the following relations are valid for $[t_0, t_1]$

$$\frac{d\lambda}{dt} = -\left(\frac{\partial H}{\partial x}\right)^T - \left(\frac{\partial \varkappa}{\partial x}\right)^T \beta = -\left(\frac{\partial H_1}{\partial x}\right)^T; \tag{3.98}$$

$$\frac{dx}{d} = \left(\frac{\partial H_1}{\partial \lambda}\right)^T = \left(\frac{\partial H}{\partial \lambda}\right)^T; \tag{3.99}$$

$$\beta_j \varkappa_j = 0, \quad (j = 1, 2, \ldots, \nu_1), \tag{3.100}$$

where

$$\beta \leqslant 0. \tag{3.101}$$

For all fixed (t, x, λ) and u, which satisfy (3.92), the maximum principle is satisfied (see Section 3.4.3)

$$H(t, x, \lambda, u^*) \leqslant H(t, x, \lambda, u), \qquad (3.102)$$

i.e.,

$$\min_{u \in U^m} H(t, x, \lambda, u) = H(t, x, \lambda, u^*),$$

where Hamiltonian H is determined, as in Section 3.4.2, by the expression

$$H = \lambda_0 f_0 + \lambda^T f, \qquad (3.103$$

and

$$H_1 = H + \beta^T \varkappa. \qquad (3.104)$$

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If minimum H is achieved in an interior point of region U^m,

$$\frac{\partial H_1}{\partial u} = \frac{\partial H}{\partial u} + \left(\frac{\partial \varkappa}{\partial u}\right)^T \beta. \qquad (3.105)$$

At cusps t_α the following conditions are satisfied:

a) conjugate vector $\lambda(t)$ is continuous, i.e.,

$$\lambda(t_\alpha + 0) = \lambda(t_\alpha - 0); \qquad (3.106)$$

b) the function H is continuous, i.e.,

$$H(t_\alpha, x(t_\alpha), \lambda(t_\alpha), u^*(t_\alpha + 0)) = H(t_\alpha, x(t_\alpha), \lambda(t_\alpha), u^*(t_\alpha - 0)) \qquad (3.107)$$

(condition (3.102) is satisfied with the equal sign);

c) equations (3.100) and (3.105) remain valid.

Conditions "a," "b," "c" are analogous to the Weierstrass-Erdman conditions (see Section 3.9.2).

At the end point (t_1, x_1) for all dt_1, $dx(t_1)$ the conditions of transversality are satisfied

$$\left[f_0 + \lambda^T f + \frac{\partial \Phi}{\partial t_1} + \mu^T \frac{\partial q}{\partial t_1} \right]_{t=t_1} dt_1 + \left[\left(\frac{\partial \Phi}{\partial x}\right)^T + \left(\frac{\partial q}{\partial x}\right)^T \mu - \lambda \right]_{t=t_1}^T dx(t_1) = 0; \qquad (3.108)$$

$$q(t_1, x(t_1)) = 0.$$

From (3.108) it follows that

$$H(t_1)=(f_0+\lambda^{\tau}\mathbf{f})_{t_1}=-\left[\frac{\partial\Phi}{\partial t_1}+\mu^{\tau}\frac{\partial q}{\partial t_1}\right]_{t_1}; \tag{3.109}$$

$$\lambda(t_1)=\left[\left(\frac{\partial\Phi}{\partial x}\right)^{\tau}+\left(\frac{\partial q}{\partial x}\right)^{\tau}\mu\right]_{t_1}. \tag{3.110}$$

Conditions (3.109) and (3.110) are simplified for the simplest problem. Thus, for example, in case (3.96) they are of the form

$$\left.\begin{aligned} &H(t_1)=0;\\ &\lambda_i(t_1)=\mu_i \quad (i=1,\ 2,\ \ldots,\ l_2);\\ &\lambda_i(t_1)=0 \quad (i=l_2+1,\ \ldots,\ n). \end{aligned}\right\} \tag{3.111}$$

3.8.4. Analog of Clebsch's Necessary Condition

We shall denote through $\bar{\kappa}$ those components of the vector of limitations κ, which at each point of the minimizing curve $x^*(t)$, $u^*(t)$ are satisfied as equalities. Let $\bar{\beta}$ be the vector of factors corresponding to them. Then /244

$$\overline{H}_1=H+\bar{\beta}^{\tau}\bar{\kappa} \tag{3.112}$$

and for the interior points of region U^m the inequality

$$\eta^{\tau}\frac{\partial^2\overline{H}_1}{\partial u^2}\eta\geqslant 0 \tag{3.113}$$

is valid on minimizing control $u^*(t)$ for all $\eta=(\eta_1,\ \eta_2,\ \ldots,\ \eta_m)^{\tau}\neq 0$, which satisfy the condition

$$\frac{\partial\bar{\kappa}}{\partial u}\eta=0. \tag{3.114}$$

Here

$$\frac{\partial^2\overline{H}_1}{\partial u^2}=\begin{bmatrix} \frac{\partial^2\overline{H}_1}{\partial u_1^2},\ \ldots,\ \frac{\partial^2\overline{H}_1}{\partial u_1\partial u_m}\\ \ldots\ldots\ldots\ldots\\ \frac{\partial^2\overline{H}_1}{\partial u_m\partial u_1},\ \ldots,\ \frac{\partial^2\overline{H}_1}{\partial u_m^2} \end{bmatrix}.$$

Conditions (3.113) and (3.114) are equivalent to the requirement of positivity of roots s of the characteristic equation

$$D(s)=\det\begin{bmatrix} \frac{\partial^2\overline{H}_1}{\partial u^2}-sE, & \left(\frac{\partial\bar{\kappa}}{\partial u}\right)^{\tau}\\ \frac{\partial\bar{\kappa}}{\partial u} & O \end{bmatrix}=0. \tag{3.115}$$

The inequality to zero of the determinant of the matrix

$$\begin{bmatrix} \frac{\partial^2 \overline{H}_1}{\partial u^2} & \left(\frac{\partial \overline{\varkappa}}{\partial u}\right)^T \\ \frac{\partial \overline{\varkappa}}{\partial u} & O \end{bmatrix} \tag{3.116}$$

at all points $x^*(t)$, $u^*(t)$ on the optimal trajectory is equivalent to the Hilbert condition (see Section 3.9.4) and in the given case designates the continuity of control $u^*(t)$. If this determinant is different from zero at each point of the extremum the problem is called nondegenerate.

Corollaries. 1. The conditions for the open kernel of region $U^m(t, x)$ (conditions (3.98)-(3.102)) mean that at all points of the trajectory at which minimum H is attained for u, $u \in U^m(x, t)$ and the rigorous inequalities

$$\varkappa_l(t, x, u) > 0 \quad (l = 1, 2, \ldots, \nu) \tag{3.117}$$

are satisfied (i.e., in the so-called open kernel of region $U^m(x, t)$) the maximum principle of Section 3.4.3 which does not take into consideration the presence of relation (3.92) is valid. Here all $\beta_i = Q(i = 1, 2, \ldots, \nu_1)$ and differential equations (3.98)-(3.99) when condition (3.102) is valid, giving $u = u(t, x, \lambda)$, have the unique solution: /245

$$\left.\begin{aligned} x_l &= x_l(t, t_0, x_0, \lambda_{l0}); \\ \lambda_l &= \lambda_l(t, t_0, x_0, \lambda_{l0}). \end{aligned}\right\} \tag{3.118}$$

In this case

$$u = u(t, t_0, x_0, \lambda_{l0}) \tag{3.119}$$

and solution of the problem of optimization is imbedded in (2n + 1) parametric families of solutions while the solution of (3.118) depends on the parameters $(t, t_0, x_{i0}, \lambda_{i0})$ at least continuously.

If there are no points of discontinuity of the function u(t) on the trajectory the solution is at least twice continuously differentiable for $(t, t_0, x_{i0}, \lambda_{i0})$.

2. If $\varkappa_i(t, x, u)$ are not explicit functions of x, conditions (3.98), (3.102) are equivalent to the maximum principle of Section 3.4.3 because in this case $U^m(x, t)$ depends only on $t: U^m = U^m(t)$.

3. Conditions for the boundary of region $U^m(x, t)$ are found in the following manner. If part of the components of the vector $\varkappa$ are satisfied during determination of minimum H for u as equalities, the unknown factors β_j

can be found from conditions (3.105). If minimum H for u is attained at an interior point of region U^m, controls u_j and factors β_j are found from conditions (3.105) and of those from (3.92) which are satisfied as equalities

$$\frac{\partial H}{\partial u}+\left(\frac{\partial \bar{\varkappa}}{\partial u}\right)^{\mathrm{T}}\tilde{\beta}=0;$$
$$\bar{\varkappa}=(t,\ \mathrm{x},\ \mathrm{u})=0. \tag{3.120}$$

From (3.120) u and $\tilde{\beta}$ are found. In this case $u = u(x, \lambda)$, $\tilde{\beta} = \tilde{\beta}(x, \lambda)$ are continuous at the connecting point only if it has no discontinuity as a function of u(t).

3.9. Elements of Classical Calculus of Variations

Problems in which equations of motion are not reduced to Cauchy form (i.e., are not written as first-order differential equations which are solved relative to derivatives)[13], and control functions u(t) appear implicitly (and if for any reason such a reduction is impossible or undesirable), can be studied by methods of the classical calculus of variations. /246

3.9.1. Boltz, Mayer and Lagrange Problems

The Boltz problem. One of the most general formulations of problems with single integrals and supplementary conditions consists in the following.

Let a class of trajectories be determined by:

1) curves x(t) with coordinates $x_i(t)$ (i = 1, 2, ..., n), $t_0 \leq t \leq t_1$;

2) parameters α_j (j = 1, 2, ..., r).

Parameters α_j can be regarded as some constant coordinates of the curve C: $z(t) = (x(t), a)^{T_j}$ in (n + r)-dimensional space, $z = (x_1, x_2, \ldots, x_n, \alpha_1, \ldots, \alpha_r)^{T}$.

Let curves (x(t), a) satisfy equations of motion (or equations of constraints, which are generally not integrable) of the form a

$$F_j=(t,\ \mathrm{x},\ \dot{\mathrm{x}},\ \mathrm{a})=0 \quad (j=1,\ 2,\ldots,\ m<n) \tag{3.121}$$

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[13] We note that from the standpoint of calculations it is always desirable to reduce the equation system to Cauchy form because effective algorithms of numerical integration have been developed especially for such system (see Chapter 4).

and the conditions

$$I_k \equiv \Phi_k(t_0, \mathbf{x}(t_0), t_1, \mathbf{x}(t_1), \mathbf{a}) + \int_{t_0}^{t_1} f_k(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{a})\,dt = 0 \quad (k = 1, 2, 3, \ldots, \varrho), \tag{3.122}$$

where

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = (\dot{x}_1, \ldots, \dot{x}_n)^T.$$

It is required to find the curve in this class of trajectories which minimizes the functional

$$J = \Phi(t_0, \mathbf{x}_0, \mathbf{x}_1, t_1, \mathbf{a}) + \int_{t_0}^{t_1} f(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{a})\,dt. \tag{3.123}$$

The Mayer problem. This problem is formally derived from the Boltz problem for $f \equiv 0$, $f_k \equiv 0$ ($k = 1, 2, \ldots, \rho$). In this case boundary conditions (3.122) become general boundary conditions whose number must be $\rho < 2n + r + 2$. If the vector of parameters a is fixed, the number of degrees of freedom σ of system of differential equations (3.121), which is equal to the difference between the number of dependent variables and the number of independent differential equations, for the Mayer problem is: $\sigma = n - m$.

The Lagrange problem. This problem derives from the Boltz problem for $\Phi \equiv 0$, $f_k \equiv 0$, $k = 1, 2, \ldots, \rho$.

Types of Relations and boundary conditions. Relations of the form (3.122) /247 for $\Phi_k = \Phi_k(\mathbf{a})$, i.e., for $\int_{t_0}^{t_1} f_k(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{a})\,dt = -\Phi_k(\mathbf{a})$, where all or part of the components of vector a are fixed, are called isoperimetric. If $f_k \equiv 0$ relations of type (3.122) give the free boundary conditions. If relations of type (3.122) have the form

$$\Phi_{k_1} \equiv x_{k_1}(t_0) - x_{k_1 0} = 0 \quad (k_1 = 1, 2, \ldots, n);$$
$$\Phi_{k_2} \equiv x_{k_2}(t_1) - x_{k_2 1} = 0 \quad (k_2 = 1, 2, \ldots, n);$$
$$\Phi_{2n+1} \equiv t_0 - t_{00} = 0, \quad \Phi_{2n+2} \equiv t_1 - t_{10},$$

where $x_{k_1 0}, \ldots, t_{10}$ are known numbers, the boundary conditions are called fixed.

If $k_1 = 1, 2, \ldots, n$; $k_2 = 1, 2, \ldots, n^1 < n$; $t_0 - t_{00} = 0$; $t_1 - t_{10} = 0$, then n^1 ends are fixed, and the remaining conditions are called free boundary conditions.

If boundary conditions $\Phi_k(t_0, t_1, x_0, x_1) = 0$ (when $f_k = 0$, $k = 1, 2, \ldots, \rho$) $\Phi_{k_1}(t_0, x_0) = 0$ can be broken down into two groups: $\Phi_{k_2}(t_1, x_1) = 0$, $k_1 = 1, 2, \ldots, \rho_1$, $k_2 = \rho_1 + 1, \ldots, \rho$, $\rho_1 < n$ and if $\Phi \equiv q(t_1, x_1) - h(t_0, x_0)$, then the problem is one with separate conditions for the ends.

General conditions (3.122) are called mixed boundary conditions.

3.9.2. First Necessary Conditions for Functional Extremum Boltz Problem

The first necessary extremum condition consists of:

- rule of Lagrange factors;
- Euler-Lagrange equations;
- Erdman-Weierstrass conditions;
- conditions of transversality.

Let minimizing curve C: {x = x(t), a} permit weak (small for x(t) and for $\dot{x}(t)$ variations of $\delta x(t) = x(t) - \tilde{x}(t)$, $\delta\dot{x}(t) = \dot{x}(t) - \dot{\tilde{x}}(t)$ in any direction compatible with relations (3.121) in space X^n, $x \in X^n$, and let functions f, f_k, Φ, Φ_k have continuous derivatives up to the third order. Then the necessary extremum conditions are formulated as follows.

The rule of Lagrange factors: there exist the functions μ_0, μ_k, $\lambda_j(t)$ and the functions

$$F = \mu_0 f + \sum_{k=1}^{\rho} \mu_k f_k + \sum_{j=1}^{m} \lambda_j(t) F_j(t, x, \dot{x}, a); \tag{3.124}$$

$$L = \mu_0 \Phi(t_0, x(t_0), t_1, x(t_1), a) + \sum_{k=1}^{\rho} \mu_k \Phi_k(t_0, x(t_0), t_1, x(t_1), a) \tag{3.125}$$

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such that factors $\mu_0 \geqslant 0$, μ_k are constant and the solution of the original problem on a conditional extremum is among the solutions of the problem on an unconditional extremum for the auxiliary functional $\bar{J} = L + \int_{t_0}^{t_1} F dt$.

Comment. It can always be assumed that $\mu_0 = 1$, except in special (anormal) cases.

Euler-Lagrange equations. Between angular points [see (3.129)] of minimizing curve C: {x = x(t), a} the Euler-Lagrange equations are satisfied:

$$\frac{d}{dt}\left(F-\sum_{i=1}^{n}\dot{x}_iF_{\dot{x}_i}\right)=F_t; \tag{3.126}$$

$$F_{x_i}-\frac{d}{dt}F_{\dot{x}_i}=0 \quad (i=1, 2, \ldots, n), \tag{3.127}$$

where $F_{x_i}=\frac{\partial F}{\partial x_i}$; $F_{\dot{x}_i}=\frac{\partial F}{\partial \dot{x}_i}$; $F_t=\frac{\partial F}{\partial t}$.

Comment. Equation (3.126) is a corollary of the others (on the condition that all $x_i(t)$ have second derivatives) and for the function of F, not explicitly containing t, reduces to the first integral:

$$F-\sum_{i=1}^{n}\dot{x}_iF_{\dot{x}_i}=C \tag{3.128}$$

which is, by virtue of (3.130), (3.131), continuous on passing through a cusp.

The solutions x(t) of the Euler-Lagrange equation are called extremals, regardless of whether they are minimizing, maximizing or saddle curves for functional J with relations (3.121), (3.122).

Erdman-Weierstrass conditions. The values $F-\sum_{i=1}^{n}\dot{x}_iF_{\dot{x}_i}$ and $F_{\dot{x}_i}(i=1, 2, \ldots, n)$ are continuous on curve C: {x = x(t), a}. In particular, if curve C has a cusp for t = t', i.e., if a discontinuity (of the first order) occurs in the derivative, even on a single component $x_i(t)$:

$$\dot{x}_i^-=\frac{dx_i(t)}{dt}\Big|_{t=t'-0}\neq\frac{dx_i(t)}{dt}\Big|_{t=t'+0}=\dot{x}_i^+, \tag{3.129}$$

the following relations are valid:

$$F_{\dot{x}_i}^-=\frac{\partial F}{\partial \dot{x}_i}\Big|_{\dot{x}_i=\dot{x}_i^-}=\frac{\partial F}{\partial \dot{x}_i}\Big|_{\dot{x}_i=\dot{x}_i^+}=F_{\dot{x}_i}^+ \quad (i=1, 2, \ldots, n); \tag{3.130}$$

and

$$F^--\sum_{i=1}^{n}\dot{x}_i^-F_{\dot{x}_i}^-=\left(F-\sum_{i=1}^{n}\dot{x}_iF_{\dot{x}_i}\right)\Big|_{\dot{x}_i=\dot{x}_i^-}= \tag{3.131}$$

$$=\left(F-\sum_{i=1}^{n}\dot{x}_iF_{\dot{x}_i}\right)\Big|_{\dot{x}_i=\dot{x}_i^+}=F^+-\sum_{i=1}^{n}\dot{x}_i^+F_{\dot{x}_i}^+.$$

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Here

$$F^- = F(t, x, \dot{x}, a)|_{\dot{x}=\dot{x}^-}; \quad F^+ = F(t, x, \dot{x}, a)|_{\dot{x}=\dot{x}^+};$$
$$\dot{x}^+ = (\dot{x}_1^+, \dot{x}_2^+, \ldots, \dot{x}_n^+)^T; \quad \dot{x}^- = (\dot{x}_1^- = \dot{x}_2^-, \ldots, \dot{x}_n^-)^T.$$

Conditions of transversality. End points 0 and 1 of curve C: {x = x(t), a} are such that the equality

$$\left[\left(F - \sum_{i=1}^{n} \dot{x}_i F_{\dot{x}_i}\right) dt + \sum_{i=1}^{n} F_{\dot{x}_i} dx_i\right]\Bigg|_0^1 + dL + \sum_{j=1}^{r} \int_{t_0}^{t_1} F_{a_j} da_j dt = 0 \tag{3.132}$$

is satisfied identically for dt_0, dt_1, $dx_{i0} = dx_i(t_0)$, $dx_{i1} = dx_i(t_1)$, da_j (i.e., for all arbitrary and independent stated variations of the ends of trajectories and variations of parameters). Here dL is the complete differential of the function $L(t_0, t_1, x(t_0), x(t_1), a, \mu_k)$:

$$dL = \frac{\partial L}{\partial t_0} dt_0 + \sum_{i=1}^{n} \frac{\partial L}{\partial x_{i0}} dx_{i0} + \frac{\partial L}{\partial t_1} dt_1 + \sum_{i=1}^{n} \frac{\partial L}{\partial x_{i1}} dx_{i1} + \sum_{j=1}^{r} \frac{\partial L}{\partial a_j} da_j. \tag{3.133}$$

Comment. If $t_0 = t_0(a)$, $t_1 = t_1(a)$, then $dt_0 = \sum_{j=1}^{r} \frac{\partial t_0(a)}{\partial a_j} da_j$, $dt_1 = \sum_{j=1}^{r} \frac{\partial t_1(a)}{\partial a_j} da_j$.

Because of the independence of dt_0, dt_1, dx_{i0}, dx_{i1} condition (3.132) is equivalent to 2n + 2 + r equations of the form

$$\left(F - \sum_{i=1}^{n} \dot{x}_i F_{\dot{x}_i} + \frac{\partial L}{\partial t}\right)_{t=t_1} dt_1 = 0, \ldots, \left(F_{\dot{x}_i} + \frac{\partial L}{\partial x_i}\right)_{t=t_1} dx_{i1} = 0 \quad (i = 1, 2, \ldots, n); \tag{3.134}$$

$$\left(F - \sum_{i=1}^{n} \dot{x}_i F_{\dot{x}_i} + \frac{\partial L}{\partial t}\right)_{t=t_0} dt_0 = 0, \ldots, \left(F_{\dot{x}_i} + \frac{\partial L}{\partial x_i}\right)_{t=t_0} dx_{i0} = 0 \quad (i = 1, 2, \ldots, n); \tag{3.135}$$

$$\left(\frac{\partial L}{\partial a_j} + \int_{t_0}^{t_1} \frac{\partial F}{\partial a_j} dt\right) da_j = 0 \qquad (j = 1, 2, \ldots, r), \tag{3.136}$$

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the number of which is sufficient so that the unknowns μ_0, μ_k (k = 1, 2, ..., ρ) $\lambda_j(t)$ (j = 1, 2, ..., m), $x_i(t)$ (i = 1, 2, ..., n), a_j (j = 1, 2, ..., r) are determined jointly through equations (3.121), (3.122), and (3.127).

3.9.3. Second Necessary Condition for the Functional Minimum in the Boltz Problem (Weierstrass Condition) for the Case $f \equiv 0$, $f_k \equiv 0$

For the permissible curve C: {x = x(t), a}, which achieves the minimum in the Boltz problem, a system of factors μ_k (k = 0, 1, ..., ρ), $\lambda_j(t)$ (j = 1, 2, ..., m) always exists such that the rule of factors (see Section 3.9.2) is obeyed for curve C with these factors, and for any element (t, x, $\dot{x}$, μ, λ) (including cusps) of curve C, the Weierstrass function E(t, x, $\dot{x}$, λ, $\dot{X}$):

$$E(t, x, \dot{x}, \lambda, \dot{X}) = F(t, x, \dot{X}, \lambda) - F(t, x, \dot{x}, \lambda) - \sum_{i=1}^{n} (\dot{X}_i - \dot{x}_i) F_{\dot{x}_i}(t, x, \dot{x}, \lambda) \tag{3.137}$$

satisfies the inequality

$$E(t, x, \dot{x}, \lambda, \dot{X}) \geqslant 0. \tag{3.138}$$

Inequality (3.138) holds for all possible permissible elements (t, x, $\dot{X}$, λ), which do not coincide with the elements (t, x, $\dot{x}$, λ) of curve C, but which satisfy the conditions

$$F_j(t, x, \dot{x}, a) = 0 \quad (j = 1, 2, \ldots, m).$$

If minimizing curve C: {x = x(t), a} is normal, factor system $\mu_0 = 1$, μ_k, $\lambda_j(t)$ (j = 1, 2, ..., m, k = 1, 2, ..., ρ) is unique and the Weierstrass condition for this system is satisfied.

3.9.4. Third Necessary Condition for Minimum in the Boltz Problem (Legendre-Clebsch Condition) for the case $f = 0$, $f_k = 0$

If the curve C: {x(t), a} achieves the minimum in the Boltz problem a factor system μ_0, μ_k (k = 1, 2, ..., ρ), $\lambda_j(t)$ (j = 1, 2, ..., m) is always found such that the rule of factors is obeyed for this curve C and the inequality

$$\sum_{l=1}^{n} \sum_{k=1}^{n} F_{\dot{x}_l \dot{x}_k}(t, x, \dot{x}, \lambda) \xi_l \xi_k \geqslant 0 \tag{3.139}$$

is satisfied for any one of its elements $(t, x, \dot{x}, \mu, \lambda)$ for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \neq (0, 0, \ldots, 0)$, which satisfy the equations /251

$$\sum_{i=1}^{n} F_{j\dot{x}_i}(t, x, \dot{x})\xi_i = 0 \quad (j = 1, 2, \ldots, m), \tag{3.140}$$

where

$$F_{jx_i} = \frac{\partial F_j}{\partial \dot{x}_i}; \quad F_{\dot{x}_i\dot{x}_k} = \frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_k}.$$

In this problem the matrix plays an important role:

$$\begin{bmatrix} F_{\dot{x}_i\dot{x}_k} & F_{\gamma\dot{x}_i} \\ F_{\alpha\dot{x}_k} & 0 \end{bmatrix} = \begin{bmatrix} F_{\dot{x}\dot{x}} & F_{\dot{x}} \\ (F_{\dot{x}})^T & 0 \end{bmatrix} \tag{3.141}$$

$$(i, k = 1, 2, \ldots, n), \quad (\alpha, \gamma = 1, 2, \ldots, m), \quad F_{\dot{x}} = \frac{\partial(F_1, F_2, \ldots, F_m)}{\partial(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)}; \quad F_{\dot{x}\dot{x}} = \left[\frac{\partial^2 F}{\partial \dot{x}_i \partial \dot{x}_k}\right].$$

The determinant of this matrix is called the Hilbert determinant. Variational problems with a non-zero Hilbert determinant are called regular (non-degenerate).

3.9.5. Fourth Necessary Condition in the Boltz Problem (Jacobi-Mayer-Kneser)

The Jacobi-Mayer-Kneser condition is non-local (integral) in nature and typifies the extremum condition of the entire curve as a whole based on examination of the behavior of extrema which lie in a small vicinity around a given extremum.

The Jacobi-Mayer-Kneser condition. In order that the extremum C: $\{x(t)\}$ impart the minimum to the functional on segment $[t_0, t_1]$ in the Boltz problem it is necessary that the segment $[t_0, t_1]$ not contain any points conjugate with t_0.

Conjugate point. It is assumed that extremum C: $\{x(t)\}$ has a point $\tilde{t}$, $t_0 < \tilde{t} < t_1$, which is conjugate with t_0 on the interval (t_0, t_1) if there indeed does exist a series of extrema which begin at the same initial point $(t_0, x(t_0))$ and which are infinitely close to a given extremum $x(t)$ such that each one of these extrema intersects the given extremum $x(t)$ and the series of points of intersection has as its limit point $\tilde{t}$. Conjugate point $(\tilde{t}, \tilde{x}(\tilde{t}))$ is the point of tangency of the extremum $x(t)$ with the envelope of the set of extrema of which the given extremum $x(t)$ is a member (note that the envelope may degenerate into a point). This indicates that at conjugate point $(\tilde{t}, x(\tilde{t}))$ the distance between a given extremum $x(t)$ and an arbitrary near-by extremum $\tilde{x}(t)$, which emerges from the same initial point $(t_0, x(t_0))$ is a value larger

than the first order of smallness in comparison with the indicated distance outside conjugate point $(\tilde{t}, x(\tilde{t}))$, i.e., when $t \leqslant t < \tilde{t}$).

The methods for determining conjugate points are quite difficult. In particular, they can be based on calculation of the Mayer-Kneser determinants. /252

For the Mayer problem (see Section 3.9.1) with fixed ends

$$F_j(t, x, \dot{x})=0 \quad (j=1, 2, \ldots, m), \qquad t_0 \leqslant t \leqslant t_1, \tag{3.142}$$

t_0, t_1 are known members,

$$x(t_0)=x_0, \ \hat{x}(t_1)=\hat{x}_1=(x_1(t_1), \ldots, x_{n-1}(t_1)), \tag{3.143}$$

x_0, $\hat{x}_1$ are known vectors,

and with the functional

$$J=\Phi(t_0, t_1, x_0, x_1)\equiv x_n(t_1) \tag{3.144}$$

conjugate point $\tilde{t}$ can be calculated as the moment of time when the Kneser determinant vanishes

$$D(\tilde{t}, \lambda_0)=\frac{\partial(x_1, x_2, \ldots, x_{n-1})}{\partial(\lambda_{10}, \lambda_{20}, \ldots, \lambda_{n-1,0})}\Bigg|_{t=\tilde{t}}= \tag{3.145}$$

$$=\begin{vmatrix} \frac{\partial x_1(t, \lambda_0)}{\partial \lambda_{10}} & \cdots & \frac{\partial x_1(t, \lambda_0)}{\partial \lambda_{n-1,0}} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_{n-1}(t, \lambda_0)}{\partial \lambda_{10}} & \cdots & \frac{\partial x_{n-1}(t, \lambda_0)}{\partial \lambda_{n-1,0}} \end{vmatrix}_{t=\tilde{t}}=0,$$

where

$$\lambda_0=(\lambda_{10}, \lambda_{20}, \ldots, \lambda_{n-1,0})^T; \tag{3.146}$$

$\hat{x}(t, \lambda_0)=(x_1(t, \lambda_0), \ldots, x_{n-1}(t, \lambda_0))$ is the extremum which satisfies given conditions (3.143) when $\lambda = \lambda_0$.

Comment. When using numerical methods for solving a boundary value problem, sometimes [for example, in the Newton method (see Section 4.2)] $(n - 1)$ auxiliary extrema $x^{n-1}(t)$, which lie in the immediate vicinity of the main extremal, and which emerge from the same point (initial) (t_0, x_0) in linear-independent directions (corresponding to linear-independent initial conditions for Lagrange factors λ_0) are calculated simultaneously with the main extremum $x(t)$. In this case it can be proved that point $\tilde{t}$ will be

conjugate with point t_0 in the above-formulated problem if the determinant at point $\tilde{t}$

$$\Delta(\tilde{t}, \lambda_0) = \left| \begin{array}{l} x_1(t) - x_1^{(1)}(t),\ x_2(t) - x_2^{(1)}(t),\ \ldots,\ x_{n-1}(t) - x_{n-1}^{(1)}(t) \\ x_1(t) - x_1^{(2)}(t),\ x_2(t) - x_2^{(2)}(t),\ \ldots,\ x_{n-1}(t) - x_{n-1}^{(2)}(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1(t) - x_1^{(n-1)}(t),\ x_2(t) - x_2^{(n-1)}(t),\ \ldots,\ x_{n-1}(t) - \\ \qquad\qquad - x_{n-1}^{n-1}(t) \end{array} \right|_{t=\tilde{t}} \tag{3.147}$$

is an infinitely small value of higher order than for $t_0 \leq t < \tilde{t}$.

3.10. Necessary Optimality Conditions in Problems with Discontinuous Phase Coordinates /253

For flight mechanics (especially for rocket dynamics) the case when finite discontinuities (discontinuities of the first kind) are permitted in the phase trajectory is important (for example, an instantaneous "drop" of mass after stage separation). The results of the following problem with a predetermined number of discontinuities and a variable magnitude of "jump" at the point of discontinuity are useful in designing multistage rockets.

3.10.1. Brief Formulation of the Problem

Let $q - 1$ be the number of intervals within which the trajectory is continuous; t_j ($j = 1, 2, \ldots, q$) are moments of time when discontinuities of the phase coordinates begin. The points t_j are considered in the general case as unknowns. The index j indicates that the functions are examined on the j-th segment of time

$$t_j \leqslant t \leqslant t_{j+1}.$$

On each j-th segment is given the system of constraints

$$\mathbf{F}^{(j)}(t,\ \mathbf{x}(t),\ \dot{\mathbf{x}}(t)) = 0, \tag{3.148}$$

where

$$\mathbf{F}^{(j)} = (F_1^{(j)},\ F_2^{(j)},\ \ldots,\ F_m^{(j)})^{\mathrm{T}};$$
$$\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n)^{\mathrm{T}};$$
$$\dot{\mathbf{x}} = (\dot{x}_1,\ \dot{x}_2,\ \ldots,\ \dot{x}_n)^{\mathrm{T}},$$

and the boundary conditions at the point of discontinuity for the functions $x_i(t)$:

$$\mathbf{g}(t_j,\ \mathbf{x}(t_r^+),\ \ \mathbf{x}(t_s^-)) = 0, \tag{3.149}$$

where

$$\mathbf{g}=(g_1,\ g_2,\ \ldots,\ g_p)^{\mathrm{T}};$$
$$j=1,\ 2,\ \ldots,\ q;$$
$$r=j;\quad 1\leqslant j\leqslant q-1;$$
$$s=j;\quad 2\leqslant j\leqslant q;$$
$$t_1<t_2<\ldots<t_j<\ldots<t_q;$$
$$p\leqslant 2(q-1)n+q.$$

It is required to minimize the functional

$$J=\Phi(t_j,|\,\mathbf{x}(t_r^+),\ \mathbf{x}(t_s^-)). \tag{3.150}$$

Comment. Here $\mathbf{x}(t_r^+)$ are right-hand limits at point of discontinuity t_j, and $\mathbf{x}(t_s^-)$ are left-hand limits.

3.10.2. Necessary Optimality Conditions /254

The necessary conditions of the functional extremum (3.150) are:

- rule of Lagrangian factors;
- Euler-Lagrange equations;
- Erdman-Weierstrass conditions;
- conditions of transversality.

For the examined discontinuity problems these conditions have the following form.

Rule of factors. The Lagrangian functions are introduced for discontinuity problems:

$$F^{(j)}=\sum_{i=1}^{m}\lambda_i F_i^{(j)}\quad (j=1,\ 2,\ \ldots,\ q-1) \tag{3.151}$$

and

$$L=\Phi+\sum_{k=1}^{p}\mu_k g_k, \tag{3.152}$$

and then the functions $x_i(t)$, $\lambda_i(t)$, μ_k are sought which satisfy (3.148), (3.149) and which impart a stationary[14] value to the auxiliary functional $\overline{J}$:

[14] The value J, whose variation δJ is equal to zero, is called a stationary quantity: $\delta J = 0$.

$$\bar{J}=L+\int_{t_1}^{t_q} F\,dt=L+\sum_{j=1}^{q-1}\int_{t_j}^{t_{j+1}} F^{(j)}\,dt. \tag{3.153}$$

In this case variation $\delta\bar{J}$ of functional $\bar{J}$ is expressed as:

$$\begin{aligned}\delta\bar{J}=&\sum_{i=1}^{n}\left[\frac{\partial L}{\partial x_i(t_1)}-\left(\frac{\partial F^{(1)}}{\partial \dot{x}_i}\right)_{t_1}\right]dx_i(t_1)+\\&+\sum_{i=1}^{n}\left[\frac{\partial L}{\partial x_i(t_2^-)}+\left(\frac{\partial F^{(1)}}{\partial \dot{x}_i}\right)_{t_2^-}\right]dx_i(t_2^-)+\\&+\sum_{i=1}^{n}\left[\frac{\partial L}{\partial x_i(t_2^+)}-\left(\frac{\partial F^{(2)}}{\partial \dot{x}_i}\right)_{t_2^+}\right]dx_i(t_2^+)+\ldots+\\&+\sum_{i=1}^{n}\left[\frac{\partial L}{\partial x_i(t_q)}+\left(\frac{\partial F^{(q-1)}}{\partial \dot{x}_i}\right)_{t_q}\right]dx_i(t_q)+\\&+\left[\frac{\partial L}{\partial t_1}+\sum_{i=1}^{n}\left(\frac{\partial F^{(1)}}{\partial \dot{x}_i}\dot{x}_i\right)_{t_1}\right]dt_1+\left[\frac{\partial L}{\partial t_2}-\sum_{i=1}^{n}\left(\frac{\partial F^{(1)}}{\partial \dot{x}_i}\dot{x}_i\right)_{t_2^-}+\right.\\&\left.+\sum_{i=1}^{n}\left(\frac{\partial F^{(2)}}{\partial \dot{x}_i}\dot{x}_i\right)_{t_2^+}\right]dt_2+\ldots+\left[\frac{\partial L}{\partial t_q}-\sum_{i=1}^{n}\left(\frac{\partial F^{(q-1)}}{\partial \dot{x}_i}\dot{x}_i\right)_{t_q}\right]dt_q+\\&+\sum_{i=1}^{n}\int_{t_1}^{t_2^-}\left[-\frac{d}{dt}\left(\frac{\partial F^{(1)}}{\partial \dot{x}_i}\right)+\frac{\partial F^{(1)}}{\partial x_i}\right]\delta x_i\,dt+\\&+\ldots+\sum_{i=1}^{n}\int_{t_{q-1}^+}^{t_q}\left[-\frac{d}{dt}\left(\frac{\partial F^{(q-1)}}{\partial \dot{x}_i}\right)+\frac{\partial F^{(q-1)}}{\partial x_i}\right]\delta x_i(t)\,dt.\end{aligned} \tag{3.154}$$

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The Euler-Lagrange equations. From expression (3.154) it follows that if x(t) is a curve which imparts a stationary value to functional $\bar{J}$ (i.e., $\delta\bar{J}=0$) the Euler-Lagrange equations (see Section 3.9.2) are satisfied between the points of the discontinuity:

$$\frac{d}{dt}\left(\frac{\partial F^{(j)}}{\partial \dot{x}_i}\right)-\frac{\partial F^{(j)}}{\partial x_i}=0 \tag{3.155}$$

$$(i=1,\ 2,\ldots,\ n;\quad j=1,\ 2,\ldots,\ q-1).$$

The Erdman-Weierstrass conditions and the conditions of transversality. The relations which generalize the conditions of transversality and the Erdman-Weierstrass conditions (see Section 3.9.2) are satisfied at end points t_1, t_q and points of discontinuity t_j:

1) when $t = t_1$

$$\frac{\partial L}{\partial t_1}+\sum_{i=1}^{n}\left[\frac{\partial F^{(1)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t=t_1}=0;\quad \frac{\partial L}{\partial x_i(t_1)}-\left[\frac{\partial F^{(1)}}{\partial \dot{x}_i}\right]_{t=t_1}=0; \tag{3.156}$$

2) when $t = t_j$ ($j = 2, 3, \ldots, q - 1$)

$$\frac{\partial L}{\partial x_i(t_j^-)}+\left[\frac{\partial F^{(j-1)}}{\partial \dot{x}_i(t)}\right]_{t=t_j^-}=0;\quad \frac{\partial L}{\partial x_i(t_j^+)}-\left[\frac{\partial F^{(j)}}{\partial \dot{x}_i(t)}\right]_{t=t_j^+}=0; \tag{3.157}$$

$$\frac{\partial L}{\partial t_j}+\sum_{i=1}^{n}\left[\frac{\partial F^{(j)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t=t_j^+}-\sum_{i=1}^{n}\left[\frac{\partial F^{(j-1)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t=t_j^-}=0; \tag{3.158}$$

3) when $t = t_q$

$$\frac{\partial L}{\partial t_q}-\sum_{i=1}^{n}\left[\frac{\partial F^{(q-1)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t_q}=0;\quad \frac{\partial L}{\partial x_i(t_q)}+\left[\frac{\partial F^{(q-1)}}{\partial \dot{x}_i}\right]_{t_q}=0. \tag{3.159}$$

For problems with fixed (known) numbers of discontinuities (jumps) boundary conditions of the type (3.149) include relations of the form /256

$$g_k\equiv x_i(t_j^-)-x_i(t_j^+)-\Delta_i^{(j)}=0, \tag{3.160}$$

where $\Delta_i^{(j)}$ is a constant (the magnitude of jump x_i at moment of time t_j), $i = 1, 2, \ldots, n$, $j = 2, 3, \ldots, q - 1$, $k = 1, 2, \ldots, p$.

Then, for $t = t_j$ ($j = 2, 3, \ldots, q - 1$) conditions (3.157) and (3.158) take the form

$$\frac{\partial L}{\partial t_j}+\sum_{i=1}^{n}\left[\frac{\partial F^{(j)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t=t_j^+}-\sum_{i=1}^{n}\left[\frac{\partial F^{(j-1)}}{\partial \dot{x}_i}\dot{x}_i\right]_{t=t_j^-}=0; \tag{3.161}$$

$$\frac{\partial L}{\partial x_i(t_j)}-\left[\frac{\partial F^{(j)}}{\partial \dot{x}_i}\right]_{t=t_j^+}+\left[\frac{\partial F^{(j-1)}}{\partial \dot{x}_i}\right]_{t=t_j^-}=0. \tag{3.162}$$

BASIC DEFINITIONS

E - estimate of local error by integration method.
H - Hamilton function.
J - quality criterion.
$f(t, x)$ - vector-function of the scalar argument t and of vector argument x (column-vector).
$f_i(t, x)$ - component of vector-function $f(t, x)$.
t - independent variable (time).
$x(t)$ - vector-function of scalar argument t, column-vector.
$x_i(t)$ - component of vector-function $x(t)$.
x - column vector.
x_i - component of vector x.
$\|x\|$ - norm of vector x.

Indices

"0" - values corresponding to initial moment of time $t = t_0$.
N, 1 - values corresponding to terminal moment of time $t = t_N = T$; $t = t_1$.
n - order of differential system equation, number of finite equations, the dimensions of vectors x or f.
r - number of integration step, iteration number, number of partition interval.
p - order of integration method, order of convergence method.

Symbols

T - symbol of transposition of vector or matrix.
$\arg\max_{u \in U} H(u)$, $\arg\min_{u \in U} H(u)$ - values of vector u which impart maximum (minimum) to function H(u) in set U.
$\in$ - symbol of membership of an element in some set.
$\{\cdot\}^{\Delta} \frac{d}{dt}$ - symbol of differentiation with respect to time argument t.
$\triangleq$ - symbol meaning: "equal by definition."
$O(\varepsilon^p)$ - symbol of infinitesimal of p-th order in relation to ε: $\lim_{\varepsilon \to 0} O(\varepsilon^p)/\varepsilon^p = C \neq 0$, and $\lim_{\varepsilon \to 0} O(\varepsilon^p)/\varepsilon^{p-1} = 0$.

The final stage of the engineering solution of flight mechanics problems is to obtain and analyze numerical results. This stage cannot be completed without using numerical methods of solving differential, algebraic and transcendental equations, and numerical methods of optimization.

From an engineering standpoint the problem is now considered solvable only if an effective method is indicated for computing the required solution with sufficient accuracy within an acceptable period of time using a limited volume of stored intermediate results. In this case the computations themselves, as a rule, are done with the aid of a rather large-volume computer. This is the difference between the current state of the art of computer mathematics and what it used to be, when because of inadequate computer hardware preference was given to calculations by explicit (analytical) equations with extensive involvement of auxiliary tables. /258

The analytical methods are still important today, but their range of application has changed somewhat.

The basic technological equipment for solving flight mechanics problems are high speed digital computers with programmed control and analogue computers.

Experience in the application of analogue and digital computers has established that the simplest flight mechanics problems, involving analyses of motion stability, systems stabilization, control, etc., should be solved with analogue computers, and analyses of flight trajectories and optimization of trajectories should be done with digital computers. The combined analysis of complex trajectory problems and problems of stabilization and control should be done with the digital computer, or with special combined computers consisting of digital and analogue computers, combining their advantages. The application of digital computers for solving complex nonlinear flight mechanics problems is now the most universally accepted method. Therefore numerical methods based principally on the use of the digital computer are described in this chapter.

The following factors must be borne in mind when selecting a specific numerical method for a given problem:

1. Required accuracy of final result and stability of numerical method or algorithm (i.e., the way an error made on a given step affects the error of the final result).

2. Capability and complexity of estimating the error in each step of the computation, which makes it possible, if need be, to alter automatically the course of the computation process in order to achieve the required accuracy.

3. Overall volume of computations required for obtaining the final result with the given accuracy. This volume depends on the complexity of computations in each step of the process and on the total number of steps; the latter, in turn, depends on the convergence rate of the algorithm. /259

4. Complexity of programming of the method in the digital computer. This is governed primarily by the complexity of the algorithm.

5. Characteristics of specific digital computer (its speed, memory capacity, read-out method, size of bit registered, etc.) in which the problem is to be solved.

When solving flight mechanics problems it is necessary to work with the following basic computer problems:

– solution of differential equations systems (problems with initial data and boundary problems);

– solution of algebraic and transcendental equations systems;

– minimization (maximization) of functions of finite number of variables;

– solution of problems of trajectory optimization and optimal control (problems of minimization or maximization of functionals).

4.1. Numerical Methods of Integrating Differential Equations of Motion

The equations of motion of aircraft are the main subject of analyses in flight mechanics.

The problems of analyzing flight trajectories, determination of orientation, analysis of motion relative to the center of mass, analysis of automatic flight control system, of guidance and navigation systems, etc. are described by differential equations systems.

In the above examples the equations of motion comprise a system of ordinary differential equations, each of which is of the first or second order and is solvable relative to the highest derivative:

$$\dot{x}_l = f_l(t, x_1, x_2, \ldots, x_n) \quad (l = 1, 2, \ldots, n) \tag{4.1}$$

or

$$\dot{x}_l = \varphi_l(t, x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n), \quad (l = 1, 2, \ldots, n) \tag{4.2}$$

In these equations the right sides f_i and φ_i are functions of the parameters of motion (of the forces of drag, gravity, weight, geometric dimensions of aircraft, etc.), usually given in graphs, tables, and sometimes in analytical form. Often data presented in graphs and tables are approximated by some analytical dependence.

The problem of integrating system (4.1), (4.2) on a given range of change of the independent variable $t_0 \leqslant t \leqslant T$ becomes defined if either the initial or boundary conditions are given.

The *initial conditions* for system (4.1) are /260

$$x_i(t_0)=x_{i0} \quad (i=1, 2, \ldots, n), \tag{4.3}$$

where x_{i0} are known values.

The initial conditions for system (4.2) are

$$\left.\begin{aligned} &x_i(t_0)=x_{i0};\\ &\dot{x}_i(t_0)=\dot{x}_{i0} \quad (i=1, 2, \ldots, n). \end{aligned}\right\} \tag{4.4}$$

where x_{i0}, $\dot{x}_{i0}$ are known values.

Problems with boundary conditions are examined in Section 4.2. The equations of motion in form (4.2) with initial conditions (4.4) can be reduced to an equation system of the form (4.1) with initial conditions (4.3) by introducing additional dependent variables

$$x_{n+i}=x_i \quad (i=1, 2, \ldots, n). \tag{4.5}$$

In these symbols system (4.2) and initial conditions (4.4) acquire the form

$$\left.\begin{aligned} &x_i=x_{n+i};\\ &\dot{x}_{n+i}=\varphi_i(t, x_1, x_2, \ldots, x_{2n}); \end{aligned}\right\} \tag{4.2'}$$

$$\left.\begin{aligned} &x_i(t_0)=x_{i0};\\ &x_{n+i}(t_0)=x_{i0} \quad (i=1, 2, \ldots, n), \end{aligned}\right\} \tag{4.4'}$$

which coincides with system (4.1), but for 2n unknowns x_i (i = 1, 2, ..., 2n).

The solution of equation (4.1) can be found in closed form after making certain, sometimes extremely important simplifications. In the vast majority of cases, when such simplifications cannot be made for some reason, the solution of equations (4.1) cannot be found as explicit expressions through elementary functions. Approximate solutions as power or trigonometric series in problems of atmospheric flight dynamics have not gained wide acceptance in connection with difficulties in analytical representation of aerodynamic forces, moments and engine thrust characteristics (they are usually represented graphically and often are not continuously differentiable functions of their arguments).

The problem of numerical integration of system (4.1) with initial conditions (4.3) on segment $[t_0,T]$ consists in selecting the partition t_0, t_1, t_2, ..., t_r, ..., t_N of segment $[t_0, T]$ into N sections and constructing a table of values of the variables x_i at the ends of these sections so that the error between the exact solution $x_i(t_r)$ at moment t_r and the approximate solution x_{ir} satisfies prescribed requirements on integration accuracy.

The difference $\Delta t_r = t_{r+1} - t_r$ of values of the argument on the ends of an elementary integration section is called the integration step. If the integration step is the same for all sections, i.e., if $\Delta t_r = \Delta t = \text{const}$ for all r, then integration is said to have a constant step; if $\Delta t_r \neq \Delta t_j$ $(r \neq j)$, then as a variable step. Here r is the number of integration step. /261

The error of numerical integration is attributed to the following factors.

— approximate character of integration methods, since some of the terms in the expressions approximating the exact solution are discarded; the resulting error is called local error, method error (round-off error);

— approximate character of arithmetic operations on numbers with a limited number of bits.

In these operations the solutions are always rounded off. The resulting error is called round-off error. Round-off errors can accumulate during the solution process. Furthermore, it should be recalled that when using the digital computer all elementary functions are computed approximately using standard subprograms:

— spreading of errors (instability of numerical integration method), leading to accumulation and build-up of method error and round-off error. In stable computation methods small errors accumulate rapidly, but in unstable methods they increase without bounds.

The various numerical integration methods exhibit different magnitudes of error. An important feature here is the error of the integration method in each step (local error method). This gives an idea of the error resulting simply from the approximate character of the method without consideration of round-off errors and can be used (together with evaluation of the complexity of the method and volume of stored information required for its execution) as the basis for the preliminary choice of method. The exact expression for method error in each step is not known for all methods; even convenient expressions for estimates of method error are often unknown. However, the order of this error (also called the order of local error) as a function of the step Δt, is known for all methods.

If $x_i(t_{r+1})$ is the exact solution of a differential equation, and $x_{i,\,r+1}$ is the approximate solution, computed in one step of length Δt using some numerical integration method and exact information about initial conditions $x_i(t_r)$ and required previous values $x_i(t_{r-1})$, $x_i(t_{r-2})$, ..., i = 1, ..., n, and /262 if here the relation

$$x_i(t_r) - x_{ir} = O(\Delta t^{p+1}), \tag{4.6}$$

is valid for error $x_i(t_r) - x_{i,\,r}$, then the number (p + 1) is called the order of method error per step (order of local error).

The numerical value of the main term of expression $O(\Delta t^{p+1})$, found by some method, is called the estimate of local error and is denoted through E. Then equation (4.6) becomes

$$x_i(t_r) - x_{ir} \approx E. \tag{4.6'}$$

The order of method error per step is closely related to the concept of order of the integration method.

A numerical integration method is called a p-th order method if the equation used in the method for x_{r+1} is equivalent to the segment of the Taylor series in powers of Δt up to the p-th order, inclusively. Thus the method order is one less than the order of local error.

Described in this chapter are numerical methods of integrating a problem with initial data (Cauchy's problem) for equation systems of the form (4.1), (4.3); the computational methods and their execution in the digital computer are also stated. In addition, special methods of integrating system (4.2), (4.4) without reducing it to a system of the form (4.2), (4.4) are described.

4.1.1. Adams' Method

Adams' method is a finite-different single-point multi-step method of numerical integration of ordinary differential equations. In such methods, in order to compute x_{r+1} at t_{r+1} it is necessary to compute the right sides f(t, x) at only one point; here the previous values of x and f(t, x), computed in several preceding integration steps, are used.

Initial data. Let there be given a system of ordinary first-order differential equations in standard Cauchy form:

— in scalar form

$$\frac{dx_l}{dt} = f_l(t, x_1, x_2, \ldots, x_n) \quad (l = 1, 2, \ldots, n); \tag{4.7}$$

— in vectorial form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \tag{4.7'}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}. \tag{263}$$

Initial conditions for $t = t_0$;

— in scalar form

$$x_l(t_0) = x_{l0} \quad (l = 1, 2, \ldots, n); \tag{4.8}$$

— in vectorial form

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{4.8'}$$

Required: to find the approximate solution of this system on segment $[t_0, T]$.

The idea of the method is to express the desired function x_{r+1} at point t_{r+1} through linear combinations of functions $\dot{x}_{r-s}$, $\dot{x}_{r-s+1}$, ..., $\dot{x}_{r-1}$, $\dot{x}_r$, x_r at points t_{r-s}, t_{r-s+1}, ..., t_{r-1}, t_r (extrapolation equations) or through linear combinations of functions $\dot{x}_{r-s+1}$, $\dot{x}_{r-s+2}$, ..., $\dot{x}_r$, $\dot{x}_{r+1}$, x_r at points t_{r-s+1}, t_{r-s+2}, ..., t_r, t_{r+1} (interpolation equations). Both types of equations are the corollary of Newton's equation for reverse interpolation.

Calculation equations. The equations presented below are written for one scalar equation of the system (4.7). For integration of the entire system (4.7) these equations must be applied to each component. Extrapolation equations are

used in practical calculations. Interpolation equations are used only in combination with extrapolation equations (see Section 4.1.2).

Adams' Extrapolation Equation[15]. This equation is expressed through differences $\nabla^p f_r$ or ordinates $f_r, \ldots, f_{r-s}$:

– difference equation

$$x_{r+1}=x_r+\Delta t\sum_{p=0}^{s}\beta_p\nabla^p f_r; \tag{4.9}$$

– ordinate equation

$$x_{r+1}=x_r+\Delta t\sum_{p=0}^{s}\alpha_{sp}f_{r-p}, \tag{4.10}$$

where $x_r=x(t_r)$;

$\Delta t=t_{r+1}-t_r$ is the integration step;

$$f_r=f(t_r, x_r);$$
$$\nabla^0 f_r=f_r;$$

$\nabla f_r=f_r-f_{r-1}$ is the left first-order difference and $\nabla^p f_r=\nabla(\nabla^{p-1}f_r)$ /264 is the left p-th order difference;

$$\beta_p=\frac{1}{p!}\int_0^1\xi(\xi+1)\ldots(\xi+p+1)d\xi;$$

α_{sp} are coefficients satisfying the condition

$$\sum_{p=0}^{s}\alpha_{sp}=1 \quad (p=0, 1, 2, \ldots, s).$$

The number p = s + 1 defines the order of Adams' equation (method).

Equations of order $p \leqslant 5$ are most often used in practice.

The coefficients β_p and α_{sp} are presented in Tables 4.1 and 4.2.

[15]This is sometimes called the Adams-Bashfort equation.

TABLE 4.1. COEFFICIENTS β_p FOR ADAMS' EXTRAPOLATION DIFFERENCE EQUATION (4.9)

p	0	1	2	3	4	5
β_p	1	1/2	5/12	3/8	251/720	95/288

TABLE 4.2. COEFFICIENTS α_{sp} FOR ADAMS' EXTRAPOLATION ORDINATE EQUATION (4.10)

p	0	1	2	3	4	5
α_{1p}	3/2	—1/2	—	—	—	—
α_{2p}	23/12	—16/12	5/12	—	—	—
α_{3p}	55/24	—59/24	37/24	—9/24	—	—
α_{4p}	1901/720	—2774/720	2616/720	—1274/720	251/720	—
α_{5p}	4277/1440	—7923/1440	9982/1440	—7298/1440	2877/1440	—475/1440

Example 1. For s = 3 (four-step fourth-order Adams' method) the difference equation (4.9) and ordinate equation (4.10) acquire the form: /265

$$x_{r+1} = x_r + \Delta t\left(f_r + \frac{1}{2}\nabla f_r + \frac{5}{12}\nabla^2 f_r + \frac{3}{8}\Delta^3 f_r\right); \tag{4.9'}$$

$$x_{r+1} = x_r + \Delta t\left(\frac{55}{24}f_r - \frac{59}{24}f_{r-1} + \frac{37}{24}f_{r-2} - \frac{9}{24}f_{r-3}\right). \tag{4.10'}$$

Adams' interpolation equation[16]. This equation, expressed through differences $\nabla^p f_r$ or ordinates $f_r \ldots, f_{r+1-s}$, is:

[16]This is sometimes called the Adams-Multon equation.

— difference equation

$$x_{r+1}=x_r+\Delta t\sum_{p=0}^{s}\beta_p^*\nabla^p f_{r+1};\tag{4.11}$$

— ordinate equation

$$x_{r+1}=x_r+\Delta t\sum_{p=0}^{s}\alpha_{sp}^*f_{r+1-p},\tag{4.12}$$

where

$$\beta_p^*=\frac{1}{p!}\int_0^1(\xi-1)\xi(\xi+1)\ldots(\xi+p-2)d\xi;$$

α_{sp}^* are coefficients satisfying condition

$$\sum_{p=0}^{s}\alpha_{sp}^*=1.$$

Some of the values of coefficients β_p^* and α_{sp}^* most frequently required for calculations are presented in Table 4.3 and 4.4.

TABLE 4.3. COEFFICIENTS β_p^* FOR ADAMS' INTERPOLATION DIFFERENCE EQUATION (4.11)

p	0	1	2	3	4	5	6
β_p^*	1	—1/2	—1/12	—1/24	—19/720	—3/160	—863/60480

Computations by equations (4.11) and (4.12) require knowledge of f_{r+1}, which is the function x_{r+1}. Thus relations (4.11) and (4.12) are essentially nonlinear equations relative to unknown x_{r+1}. Therefore integration by equations (4.11) and (4.12) is said to constitute the implicit computation method. The value of x_{r+1} obtained from the extrapolation equation (see Section 4.1.2 concerning the combined use of the extrapolation and interpolation equations) can be used for solving equation (4.12) as the initial approximation.

TABLE 4.4. COEFFICIENTS α^*_{sp} FOR ADAMS' INTERPOLATION ORDINATE EQUATION (4.12)

p	0	1	2	3	4	5
α^*_{1p}	1/2	1/2	—	—	—	—
α^*_{2p}	5/12	8/12	—1/12	—	—	—
α^*_{3p}	9/24	19/24	—5/24	1/24	—	—
α^*_{4p}	251/720	646/720	—264/720	106/720	—19/720	—
α^*_{5p}	475/1440	1427/1440	—798/1440	482/1440	—173/1440	27/1440

The coefficients β_p and β^*_p, β_p and α^*_{s0} are coupled by the relations:

$$\left.\begin{aligned}&\beta_p+\beta^*_{p+1}=\beta_{p+1};\\&\beta_n=\alpha^*_{s0}.\end{aligned}\right\} \qquad (4.13)$$

Example 2. For s = 3 (four-step fourth-order Adams' method) difference equation (4.11) and ordinate equation (4.12) are: /266

$$x_{r+1}=x_r+\Delta t\left(f_{r+1}-\frac{1}{2}\nabla f_{r+1}-\frac{1}{12}\nabla^2 f_{r+1}-\frac{1}{24}\nabla^3 f_{r+1}\right); \qquad (4.11')$$

$$x_{r+1}=x_r+\Delta t\left(\frac{9}{24}f_{r+1}+\frac{19}{24}f_r-\frac{5}{24}f_{r-1}+\frac{1}{24}f_{r-2}\right). \qquad (4.12')$$

Computational method of extrapolation. For computations by Adams' extrapolation equations it is necessary to determine beforehand the initial ("starting") x_r and f_r (r = 0, 1, 2, ..., s). These values are usually found by a different approximation method, for instance by the Runge-Kutta method (see Section 4.1.3).

In *manual calculations* it is helpful to compute with difference equations (4.9).

Ordinate equations (4.10) are usually employed for computer integration with the digital computer. /267

The computational methods based on the interpolation equations are discussed in Section 4.1.2.

Accuracy of method. Equations (4.9)-(4.12) make it possible to compute the approximate $\bar{x}_{r+1}$ of the solution $x(t_{r+1})$ of system (4.7). The error of these equations per integration step (local error) is determined by the remainder R_{s+1}.

The expressions for the remainder are:

1) for the extrapolation equation

$$R_{s+1}=\beta_{s+1}\Delta t^{s+2}x^{(s+2)}(\xi),\quad t_{r-s}\leqslant\xi\leqslant t_{r+1}; \tag{4.14}$$

2) for the interpolation equation

$$R^{*}_{s+1}=\beta^{*}_{s+1}\Delta t^{s+2}x^{(s+2)}(\eta),\quad t_{r-s}\leqslant\eta\leqslant t_{r+1}. \tag{4.15}$$

The total error of integration by Adams' method (with constant step Δt) on the entire segment $[t_0, T = t_N]$ is given by the equation

$$|\tilde{x}_r-x(t_r)|\leqslant\frac{G}{1-|\beta\Delta t|M}\left[(1+|\beta\Delta t|M)\theta+\right.$$
$$\left.+t_r\left(\frac{\delta}{\Delta t}+K\Delta t^{s+1}\right)\right]e^{\alpha\tilde{M}t_r}, \tag{4.16}$$

where $\tilde{x}_r$ ($r = 0, 1, 2, \ldots, N - 1$) is the solution of the difference equation

$\tilde{x}_{r+1}=\tilde{x}_r+\Delta t\sum_{p=0}\alpha_{sp}f_{r-p}+\delta_r$ for the extrapolation method and of the equations

$\tilde{x}_{r+1}=\tilde{x}_r+\Delta t\sum_{p=0}^{s}\alpha^{*}_{sp}f^{\cdot}_{r-p+1}+\delta_r$ for the interpolation method;

$$\beta=\begin{cases}\alpha_{s0} & \text{— for the extrapolation method,}\\ \alpha^{*}_{s0} & \text{— for the interpolation method;}\end{cases}$$

$\delta = \max \delta_r$ is the estimate of round-off error;

δ_r is the maximum round-off error in the r-th step;

θ is the maximum error of initial conditions and starting points;

G, M, $\tilde{M}$, K are constants, independent of Δt.

As seen from this equation, as Δt decreases the error first decreases due to the term $K\Delta t^{s+1}$ (reduction of method error), and then begins to increase due to $\delta/\Delta t$ (accumulation of round-off errors).

The remainder is estimated through the following expressions:

– for extrapolation equation (4.10) /268

$$|R_{s+1}| \leqslant \Delta t^{s+2} |\beta_{s+1}| |f^{s+1}|_{\max}; \tag{4.17}$$

– for interpolation equation (4.12)

$$|R^*_{s+1}| \leqslant \Delta t^{s+2} |\beta^*_{s+1}| |f^{s+1}|_{\max}, \tag{4.18}$$

where $|f^{s+1}|_{\max}$ is the estimate of the maximum partial derivatives of order $s + 1$.

Interpolation equations (4.11) and (4.12) have less error than extrapolation equations (4.9) and (4.10) due to coefficients β^*_{s+1}, which are smaller than β_{s+1}.

Comment. The above accuracy estimates presume that the solution x(t) of the differential equations is a (s + 2)-times continuously differentiable function of time. This places certain limitations on the differentiability of the right sides f(t, x) of differential equations (4.7), which should have continuous partial derivatives of t and x up to the (s + 1)-th order. In flight mechanics problems the right sides sometimes do not satisfy these conditions. In such cases the correct use of the method requires that the "starting" points be converted during the calculation process on passage through the boundaries of violation of the stated properties. Here the point lying on the boundary of violation of smoothness is taken as the initial point (it can be determined by interpolation of points found in previous steps).

Advantages and Disadvantages of Method

1. For s = 3 and the same integration step, Adams' extrapolation method has the same order of error per step (order of local error) as the Runge-Kutta fourth-order method (see Section 4.1.3).

2. For s = 0 Adams' method corresponds to Euler's broken method (or the first-order Runge-Kutta method, cf. Section 4.1.3, Table 4.9).

3. Adams' method should be used with constant integration step Δt.

4. Adams' p-th order extrapolation method requires that a special procedure be used for determining auxiliary initial ("starting") points in order to begin the computations. The Runge-Kutta method of the same order can be used for finding them.

5. In each step of computation by Adams' extrapolation method the right side f(t, x) of the differential equation system is computed only one time.

Example 3. Numerical integration of equations of motion of aircraft by Adams' extrapolation method. /269

Formulation of problem. Determine the change of the parameters of motion by a supersonic interceptor flying in vertical plane with constant transverse load n_y = const.

Initial conditions. 1. $G(t) = G_0 = 15{,}500$ kg is the force of gravity (weight) of the aircraft.

2. $S = 50.0\ m^2$ is wing area.

3. $n_y(t) = n_{y0} = 1.8$ is transverse load.

4. The aerodynamic characteristics of the aircraft, given by relations valid in the range 11 km $\leqslant h \leqslant$ 25 km of altitudes and range $0.9 \leqslant M \leqslant 3.2$ of Mach numbers are:

$$c_x = c_{x0} + B_a c_y^{\alpha} \alpha^2; \quad B_a = 0.95 \text{ rad}^{-1};$$

$$c_y^{\alpha} = 3.10 - 0.56\ M \text{ rad}^{-1}; \quad c_{x0} = \frac{0.0034M + 0.0055}{M - 0.61};$$

$M = v/a$; $a = 295.07$ m/sec is the speed of sound.

5. The force of gravity, given by relations valid in the range 11 km $\leqslant h \leqslant$ 25 km of altitudes and range $0.9 \leqslant M \leqslant 3.2$ of Mach numbers, is:

$$P = P_{11}(M)\frac{\varrho(h)}{\varrho_{11}};$$

$$P_{11}(M) = 10650\,[0.60 + 0.55\,(M - 1.45)]\ \text{kg} =$$

$$= 104\,441\,[0.60 + 0.55\,(M - 1.45)]\ \text{n};$$

$$\varrho_{11} = 3.7204 \cdot 10^{-2}\ \text{kg} \cdot \text{sec}^2/\text{m}^4 = 3.6485 \cdot 10^{-1}\ \text{n} \cdot \text{sec}^2/\text{m}^4;$$

$\rho(h)$ is the density of the atmosphere at altitude h, corresponding to the table of standard atmosphere SA-64, GOST [State Standard] 4401-64.

Required: to find changes of flight velocity v(t), angle of inclination of trajectory $\theta(t)$ and flight altitude h(t) on segment $0 \leqslant t \leqslant 40$ sec.

Assumptions. 1. The aircraft is regarded as a point of constant mass ($G = G_0 = \text{const}$), flying in the Earth's homogeneous gravitational field in the vertical plane ($g = g_0 = 9.80665\ \text{m/sec}^2$, $\gamma_v = 0$, $\psi_v = 0$, $\beta = 0$).

2. The curvature and rotation of the Earth may be disregarded.

3. The angle of attack is small ($\sin\alpha \approx \alpha$, $\cos\alpha \approx 1$).

Equations of motion. In accordance with the assumptions made above the motion of the center of mass of the aircraft in aircraft-centered Cartesian vertical-velocity coordinate system $Ox_v y_v z_v$ during flight in the vertical plane is described by the following differential equations, reduced to standard form:

$$\left.\begin{aligned} \dot{v} &= \frac{g_0}{G_0}\,[P - Q(M, h, \alpha) - G_0 \sin\theta]; \\ \dot{\theta} &= \frac{g_0}{v}\,(n_{y0} - \cos\theta); \\ \dot{h} &= v\sin\theta, \end{aligned}\right\} \qquad \text{(I)}$$

where $Q(M, h, \alpha) = c_x qS$ is drag;

$q = 1/2\ \rho v^2$ is the velocity head (in kg/m^2, if ρ is measured in $\text{kg sec}^2/\text{m}^4$ and in n/m^2 if ρ is measured in $\text{n sec}^2/\text{m}^4$);

$\alpha = G_0 n_{y0}/P + c_y^{\alpha} qS$ is the angle of attack in radian.

Initial conditions. For $t = 0$, $v_0 = 500$ m/sec, $h_0 = 12{,}000$ m, $\theta_0 = 8°$.

Numerical solution. Numerical integration of equation system (I) with given initial conditions was accomplished with a BESM [High-speed electronic] digital computer with a constant integration step by Adams' extrapolation equation (4.10), which was applied to each equation of system (I). The following symbols can be used for reducing system (I) to form (4.1): /270

$$\left.\begin{aligned} x_1 &= v, \quad x_2 = \theta, \quad x_3 = h, \\ f_1(x_1, x_2, x_3) &= \frac{g_0}{G_0}\,[P - Q(M, h, \alpha) - G_0 \sin\theta], \\ f_2(x_1, x_2) &= \frac{g_0}{v}\,(n_{y0} - \cos\theta), \\ f_3(x_1, x_2) &= v\sin\theta. \end{aligned}\right\} \qquad \text{(II)}$$

The results of integration of aircraft motion equation system (I) by Adams' extrapolation equation (with step $\Delta t = 1$ sec) are summarized in Table 4.5. (results are given in the eight significant figures produced by the computer).

TABLE 4.5. RESULTS OF INTEGRATION OF EQUATIONS OF MOTION BY ADAMS' EXTRAPOLATION EQUATION IN DIGITAL COMPUTER WITH $\Delta t = 1$ sec.

t, sec	Parameters of motion			Comments
	v, m/sec	θ°	h, m	
0	500,00000	8,0000000	12000,000	1. Initial ("starting") points obtained by standard fourth order Runge-Kutta method with step Δt = 1 sec.
4	500,38731	11,659288	12341,576	
5	499,95167	12,583011	12446,601	
10	494,22870	17,291847	13086,942	
20	462,17631	27,539044	14903,576	
40	266,87068	59,563744	19792,786	2. Numerical results printed out through time interval Δt_{po} = = 1 sec.

Note: Commas indicate decimal points.

4.1.2. Forecasting and Correction Methods

Forecasting and correction methods are single point multi-step methods of numerical integration of ordinary differential equations. To compute x_{r+1} by these methods the right side f(t, x) need be computed at one point only, for which purpose previous values of x and f(t, x) computed in several preceding integration steps, are used.

Initial data. Let be given a system of ordinary first order differential equations, solvable in terms of the derivative:

--in scalar form /271

$$\frac{dx_l}{dt} = f_l(t, x_1, x_2, \ldots, x_n) \quad (l = 1, 2, 3, \ldots, n); \tag{4.19}$$

--in vectorial form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \tag{4.19'}$$

where

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad \mathbf{f}=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

The initial conditions for $t = t_0$ are:

--in scalar form

$$x_i(t_0)=x_{i0} \quad (i=1, 2, \ldots, n); \tag{4.20}$$

--in vectorial form

$$\mathbf{x}(t_0)=\mathbf{x}_0. \tag{4.20'}$$

Required: To find the approximate solution of the system on segment $[t_0, T]$.

The idea of the method is to compute by forecast (extrapolation) equations the value x^{pr}_{r+1} at point $t = t_{r+1}$. This value is then corrected by correction equations (interpolation equations). Special modifying equations which do not require new computations of the right side of the equations, but which are based on the use of the results of previous steps, are often used for increasing the accuracy of the solution. The equations of the correction method can be use repeatedly, i.e., in iteration[17].

Calculation equation. The equations presented below are written for one (scalar) equation of system (4.19). To integrate the entire system (4.19) these equations must be used for each component of vector x. Forecasting and correction methods incorporating various extrapolation and interpolation equations and equations which correct (modify) the predicted values, are presented in Table 4.6.

Computing method. In order to use the integration method based on forecasting and correction it is first necessary to determine the auxiliary initial (starting) x_r and $f_r (r = 0, 1, 2, \ldots, s)$.

[17] In practice, as a rule, correction equations are used once or twice.

TABLE 4.6. CALCULATION EQUATIONS OF FORECASTING AND CORRECTION METHODS (WITH AND WITHOUT MODIFIER) FOR INTEGRATING ORDINARY FIRST ORDER DIFFERENTIAL EQUATIONS $\dot{x} = f(t, x)$

Method	Forecasting and Correction Equations	Final Equations	Order of Local Error
Euler's improved ("with conversion") method	$x_{r+1}^{pr} = x_{r-1} + 2\Delta t \cdot f_r$ $x_{r+1}^{cor} = x_r + \frac{1}{2}\Delta t\,(f_{r+1}^{pr} + f_r)$	$x_{r+1} = x_{r+1}^{cor}$ or more accurately $x_{r+1} = x_{r+1}^{cor} + \frac{1}{5}(x_{r+1}^{pr} - x_{r+1}^{cor})$	$O(\Delta t^3)$
Adams-Bashfort-Moulton method	$x_{r+1}^{pr} = x_r + \frac{1}{24}\Delta t(55f_r - 59f_{r-1} + 37f_{r-2} - 9f_{r-3})$ $x_{r+1}^{cor} = x_r + \frac{1}{24}\Delta t\,(9f_{r+1}^{pr} + 19f_r - 5f_{r-1} + f_{r-2})$	$x_{r+1} = (x_{r+1}^{cor})_m$, where m is the number of computations x_{r+1}^{cor} corresponding to the condition $\|x_{r+1}^{pr} - (x_{r+1}^{cor})_m\| < \varepsilon$ or defined a priori	$O(\Delta t^5)$
Adams-Bashfort-Moulton method with modifier	$x_{r+1}^{pr} = x_r + \frac{1}{24}\Delta t(55f_r - 59f_{r-1} + 37f_{r-2} - 9f_{r-3})$ $x_{r+1}^{mod} = x_{r+1}^{pr} + \frac{251}{270}(x_r^{cor} - x_r^{pr})$ $x_{r+1}^{cor} = x_r + \frac{1}{24}\Delta t\,(9f_{r+1}^{mod} + 19f_r - 5f_{r-1} + f_{r-2})$	$x_{r+1} = x_{r+1}^{cor} - \frac{19}{270}(x_{r+1}^{cor} - x_{r+1}^{pr})$	$O(\Delta t^5)$
Hamming's method	$x_{r+1}^{pr} = x_{r-3} + \frac{4}{3}\Delta t\,(2f_r - f_{r-1} + 2f_{r-2})$ $x_{r+1}^{cor} = \frac{1}{8}(9x_r - x_{r-2}) +$ $+ \frac{3}{8}(f_{r+1}^{pr} + 2f_r - 2f_{r-1})\Delta t$	$x_{r+1} = (x_{r+1}^{cor})_m$, where m is the number of computations x_{r+1}^{cor} corresponding to the condition $\|x_{r+1}^{pr} - (x_{r+1}^{cor})_m\| < \varepsilon$ or defined a priori	$O(\Delta t^5)$

TABLE 4.6. (CONTINUED)

Method	Forecasting and Correction Equations	Final Equations	Order of Local Error
Hamming's method with modifier	$x_{r+1}^{pr}=x_{r-3}+\frac{4}{3}\Delta t(2f_r-f_{r-1}+2f_{r-2})$ $x_{r+1}^{mod}=x_{r+1}^{pr}+\frac{112}{121}(x_r^{cor}-x_r^{pr})$ $x_{r+1}^{cor}=\frac{1}{8}(9x_r-x_{r-2})+$ $+\frac{3}{8}\Delta t(f_{r+1}^{mod}+2f_r-f_{r-1})$	$x_{r+1}=x_{r+1}^{cor}-$ $-\frac{9}{121}(x_{r+1}^{cor}-x_{r+1}^{pr})$	$O(\Delta t^5)$
Milne's method	$x_{r+1}^{pr}=x_{r-3}+\frac{4}{3}\Delta t(2f_r-f_{r-1}+2f_{r-2})$ $x_{r+1}^{cor}=x_{r-1}+\frac{1}{3}\Delta t(f_{r+1}^{pr}+4f_r+f_{r-1})$	$x_{r+1}=(x_{r+1}^{cor})_m$, where m is the number of computations x_{r+1}^{cor} corresponding to the condition $\|x_{r+1}^{pr}-(x_{r+1}^{cor})_m\|<\varepsilon$ or defined a priori	$O(\Delta t^5)$
Milne's method with modifier	$x_{r+1}^{pr}=x_{r-3}+\frac{4}{3}\Delta t(2f_r-f_{r-1}+2f_{r-2})$ $x_{r+1}^{mod}=x_{r+1}^{pr}+\frac{28}{29}(x_r^{cor}-x_r^{pr})$ $x_{r+1}^{cor}=x_{r-1}+\frac{1}{3}\Delta t(f_{r+1}^{mod}+4f_r+f_{r-1})$	$x_{r+1}=x_{r+1}^{cor}-$ $-\frac{1}{29}(x_{r+1}^{cor}-x_{r+1}^{pr})$	$O(\Delta t^5)$

Definitions: $f_{r+1}^{pr}=f(t_{r+1}, x_{r+1}^{pr})$; $f_r=f(t_r, x_r)$; $f_{r+1}^{mod}=f(t_{r+1}, x_{r+1}^{mod})$.

Indices: pr, forecast; mod, modification; cor, correction

These values are usually found by single-step multi-point Runge-Kutta type methods (see Section 4.1.3.) The accuracy of computation of the starting point should not be less than that of subsequent points. The forecasting and correction methods should be used with the computer. The equations of the forecasting and correction methods can be used both with and without modifying equations. /274

Equations of low-order methods (for instance, improved Euler methods with computation of starting point by the second order Runge-Kutta method) are usually employed for manual calculations.

Accuracy of method. Estimates of local errors of various methods of integrating forecasting and correction, and the orders of these errors are presented in Tables 4.6 and 4.7, respectively.

TABLE 4.7. LOCAL ERROR OF FORECASTING AND CORRECTION METHODS.

Method	Equation for estimating local errors	Order of local error
Adams-Bashford-Moulton	$E = -\frac{19}{270}(x_{r+1}^{cor} - x_{r+1}^{pr})$	$O(\Delta t^5)$
Hamming	$E = -\frac{9}{121}(x_{r+1}^{cor} - x_{r+1}^{pr})$	$O(\Delta t^5)$
Milne	$E = -\frac{1}{29}(x_{r+1}^{cor} - x_{r+1}^{pr})$	$O(\Delta t^5)$
Improved Euler	$E = -\frac{1}{5}(x_{r+1}^{pr} - x_{r+1}^{cor})$	$O(\Delta t^3)$

Comment. The order of local error of the forecasting and correction method and its estimate are valid if the right sides $f(t, x)$ of the differential equations have a sufficiently large number of continuous partial derivatives in terms of t and x (not lower than the order of the method). In flight mechanics problems the right sides of the differential equations often do not exhibit sufficient smoothness. In such cases its points of discontinuity of smoothness must be determined (preferably quite accurately), for example, by interpolation and, starting at this point as the initial point, the calculation must be done over again (i.e., new "starting" points are found, etc.). /275

Choice of integration step. The local error, given in Table 4.7, is usually employed as a measure of the correctness of the selected integration step. If the resulting current error E is not satisfactory (i.e., $E > \varepsilon$, where ε is the prescribed integration accuracy per step), then the step is usually cut in half. If $E < \varepsilon/32$, then the step may be doubled. If the step is changed the "starting" points must be recomputed. Recomputation of the starting points can be avoided by storing twice as many points x_r in the computer memory than is required in the method (for doubling the step), and interpolation equations can be used (for decreasing the step). /275

The values of x for intermediate points (when the step is cut in half) should be computed with the following fourth-order interpolation equations:

$$\left.\begin{aligned} x_{r-\frac{1}{2}} &= \frac{1}{128}[45x_r + 72x_{r-1} + 11x_{r-2} + \\ &\quad + \Delta t(-9f_r + 36f_{r-1} + 3f_{r-2})]; \\ x_{r-\frac{3}{2}} &= \frac{1}{128}[11x_r + 72x_{r-1} + 45x_{r-2} - \\ &\quad - \Delta t(3f_r + 36f_{r-1} - 9f_{r-2})]. \end{aligned}\right\} \tag{4.21}$$

Equations (4.21) do not use all the stored information. For greater accuracy the interpolation can be accomplished with the following equations: /276

$$\left.\begin{aligned} x_{r-\frac{1}{2}} &= \frac{1}{256}[80x_r + 135x_{r-1} + 40x_{r-2} + x_{r-3} + \\ &\quad + \Delta t(-15f_r + 90f_{r-1} + 15f_{r-2})]; \\ x_{r-\frac{3}{2}} &= \frac{1}{256}[12x_r + 135x_{r-1} + 108x_{r-2} + x_{r-3} + \\ &\quad + \Delta t(-3f_r - 54f_{r-1} + 27f_{r-2})]. \end{aligned}\right\} \tag{4.22}$$

Comments. 1. When finding the starting points by one of the Runge-Kutta methods it is advisable to choose from among them the method requiring the least computer memory and in which the computations are done cyclically. The Runge--Kutta-Hill method satisfies these requirements (see Section 4.1.3).

2. Determination of the starting points should be organized such that the size of the initial integration step can be selected if it is not known apriori. Such choice of step can be made on the basis of double calculation with steps Δt and $\Delta t/2$ for determining x at point $t_0 + \Delta t$.

If

$$\| x_{\Delta t} - x_{2\frac{\Delta t}{2}} \| < \varepsilon,$$

($x_{\Delta t}$ is the solution at point $t_0 + \Delta t$ when integrating with step Δt;

$x_{2\frac{\Delta t}{2}}$ is the solution at point $t_0 + \Delta t$ when integrating with step $\Delta t/2$;

Δt is a trial initial step; ε is the prescribed accuracy; $\|...\|$ is the vector norm), then it is assumed that step Δt is suitable for the given accuracy ε and $\Delta t_0 = \Delta t/4$ can be used (with some margin) as the initial step.

3. During integration by forecasting the correction methods in cases when computation by the interpolation equations is done by iteration, it is advisable to check the accuracy against the criterion

$$\| x_{r+1}^{cor} - x_{r+1}^{pr} \| < \varepsilon, \tag{4.23}$$

where x_{r+1}^{cor} is the solution at point t_{r+1}, found from the interpolation equation (corrector); x_{r+1}^{pr} is the solution at the same point, found from the extrapolation equation (forecasting equation).

If condition (4.23 is satisfied the step is assumed to be satisfactory. If (4.23) is not satisfied the step must be changed. The procedure for changing the step is the following:

$$\Delta t_r = \begin{cases} \Delta t_{r-1}, & \text{if } \|x_{r+1}^{cor} - x_{r+1}^{pr}\| < \varepsilon \text{ or if conditions } \|x_{r+1}^{cor} - x_{r+1}^{pr}\| < \kappa_1 \varepsilon; \\ & \|(x_{r+1}^{cor})_2 - x_{r+1}^{cor}\| < \varepsilon \\ & \text{are satisfied simultaneously.} \\ \frac{1}{2}\Delta t_{r-1}, & \text{if } \|x_{r+1}^{cor} - x_{r+1}^{pr}\| > \kappa_1 \varepsilon. \\ 2\Delta t_{r-1}, & \text{if condition } \|x_{r+1}^{cor} - x_{r+1}^{pr}\| < \kappa_2 \varepsilon \text{ is three times consecutively satisfied.} \end{cases}$$

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Here ε is the given integration accuracy per step; $(x_{r+1}^{cor})_2$ is the result of recomputation by the interpolation equation; κ_1, κ_2 are parameters depending on the remainder of the integration equation. When the Adams-Bashfort-Multon fourth-order method is used $\kappa_1 = 270/19$; $\kappa_2 = 1/32$; in the case of Hamming's fourth-order method $\kappa_1 = 121/9$; $\kappa_2 = 1/32$; for Milne's fourth-order method $\kappa_1 = 29$, $\kappa_2 = 1/32$).

Features of application of forecasting and correction methods using digital computer. The methods described in Table 4.6 are included within the library of standard subprograms of certain computers.

The following points must be borne in mind when using the examined methods.

1. Numerical integration methods with forecasting and correction require, as a rule, fewer cycles for computing the right sides of the equations than the Runge-Kutta multi-point methods (see Section 4.1.3) of the same order. Thus the fourth-order Runge-Kutta method requires quadruple computation of f(t, x) in each step, whereas the Adams-Bashfort-Multon method requires only double computation.

2. Forecasting and correction methods require somewhat less computer memory volume for storing the results of preceding steps than the Runge-Kutta methods of the same order.

3. For forecasting and correction methods it is easy during computation process to establish the need for changing the integration step. This procedure is based on simple explicit equations for estimating the local error. To actually change the integration step, however, it is necessary to recompute the starting points or to use special interpolation equations and increase the volume of stored information about the results of preceding steps, which increases the size and complexity of the program.

4. The use of automatic step changing procedures, as a rule, substantially decreases computer time compared with integration with a constant step (given the same integration accuracy).

5. Sometimes multiple correction equations are required, forming an /278
iteration process for determining x_{r+1}^{cor}. Such use of the method should be approached with care, since for a large Δt the process does not always converge or converges only after a large number of iterations, which requires a great deal of computer time. In the absence of convergence after two or three iterations it is advisable to increase the step or use modifying equations.

6. The use of modifying equations does not require recomputation of the right sides and increases the accuracy of the solution.

7. When discontinuities occur at certain points $t = t'$ of the right sides of the differential equations, or if their continuous differentability up to the corresponding order (in the Adams-Bashfort-Multon method, for instance, up to the $(s + 1)$-th order) is violated, it is necessary to start the computations over to maintain the order of the method, using the point of discontinuity of the properties of continuity and differentiability of the right sides as the starting point. This point can usually be found with the aid of the corresponding interpolation equations.

Example 4. Numerical integration of equations of motion of aircraft by forecasting and correction method. To illustrate the use of forecasting and

correction methods the results of numerical solution of the problem described in Section 4.1.1 (example 3) are given. The equations of motion of an aircraft are integrated in a high-speed digital computer by the Adams-Bashfort--Multon method with a constant integration step $\Delta t = 1.0$ sec. The results are summarized in Table 4.8, where they are given to eight significant digits.

TABLE 4.8. RESULTS OF INTEGRATION OF MOTION EQUATIONS BY ADAMS-BASHFORT-MULTON METHOD IN HIGH SPEED DIGITAL COMPUTER WITH STEP $\Delta t = 1$ sec

t, sec	Parameters of Motion			Comment
	v, m/sec	θ°	h, m	
0	500.00000	8.0000000	12000.000	The initial ("starting") points are found by the standard fourth-order Runge-Kutta method with step $\Delta t = 1$ sec.
4	500.38736	11.659106	12341.557	
5	499.95186	12.582643	12446.558	
10	494.23103	17.290515	13086.739	
20	462.19181	27.535343	14902.857	
40	266.95998	59.550212	19790.925	

4.1.3. Runge-Kutta Methods

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The Runge-Kutta methods are multi-point single-step methods of numerical integration of ordinary differential equation systems. In multi-point single--step methods, in order to find the approximate solution x_{r+1} at point t_{r+1}, the right side $f(t, x)$ of the differential equation system is determined at several auxiliary points of arguments t and x. Here the value of x_r, computed at one preceding point t_r, is used for determining the stated auxiliary points. The chief advantage of single-step methods is that since information of the solution at only one preceding point is required for arriving at the approximate solution at new point t_{r+1}, there is no need for special procedures for finding the solution at several preceding initial points in order to begin the integration process. Thus, to use single-step methods it is sufficient to know the initial conditions at one point.

At the same time the Runge-Kutta equations (the most important among general single-step methods) have a drawback in that the right side of the equations is computed repeatedly in one step. This may take a great deal of time in cases when the right sides are described by complex expressions.

Initial data. Ordinary differential equations of the first or second order, solved relative to the highest derivative, are given:

— in scalar form

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n); \tag{4.24}$$

$$\frac{d^2x_i}{dt^2}=\varphi_i(t,\ x_1,\ x_2,\ \ldots,\ x_n,\ \dot{x}_1,\ \dot{x}_2,\ \ldots\ \dot{x}_n)\quad (i=1,\ 2,\ \ldots,\ n); \tag{4.25}$$

– in vectorial form

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(t,\ \mathbf{x}); \tag{4.24'}$$

$$\frac{d^2\mathbf{x}}{dt^2}=\boldsymbol{\varphi}(t,\ \mathbf{x},\ \dot{\mathbf{x}}), \tag{4.25'}$$

where

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix};\quad \mathbf{f}=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix};\quad \boldsymbol{\varphi}=\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

The initial conditions for $t = t_0$ are:

– in scalar form

for equations (4.24)

$$x_i(t_0)=x_{i0}; \tag{4.26}$$

for equations (4.25)

$$x_i(t_0)=x_{i0};\quad \dot{x}_i(t_0,\ x_{i0})=\dot{x}_{i0}; \tag{4.27}$$

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– in vectorial form

for equations (4.24')

$$\mathbf{x}(t_0)=\mathbf{x}_0, \tag{4.26'}$$

for equations (4.25')

$$\mathbf{x}(t_0)=\mathbf{x}_0,\quad \dot{\mathbf{x}}(t_0,\ \mathbf{x}_0)=\dot{\mathbf{x}}_0. \tag{4.27'}$$

Required: to find the approximate solution of the systems on the segment $[t_0, T]$

The idea of the Runge-Kutta methods is to find the approximate solution of first-order differential equation (4.24') at point t_{r+1} in the form

$$x_{r+1}=x_r+\sum_{l=1}^{n_p}\gamma_l^{(p)}k_l=x_r+\Phi_f^{(p)}(t_r,\ x_r,\ \Delta t_r), \qquad (4.28)$$
$$k_1=\Delta t_r\cdot f(t_r,\ x_r);$$

where

$$k_l=\Delta t_r\cdot f\left(t_r+\alpha_l^{(p)}\Delta t_r,\quad x_r+\sum_{j=1}^{l-1}\beta_{lj}^{(p)}k_j\right) \qquad (4.29)$$
$$(l=2,\ 3,\ 4,\ \ldots,\ n_p);$$

$\Delta t_r=t_{r+1}-t_r$ is the integration step.

The coefficients $\gamma_i^{(p)}$, $\alpha_i^{(p)}$, $\beta_{ij}^{(p)}$ in equations (4.28) and (4.29) are selected such that the first few terms of expansion of the right side of equation (4.28) into a Taylor series by powers of Δt_r coincides with the same number of terms of expansion of the exact solution $x(t_r + \Delta t_r)$ by powers of Δt_r.

The Runge=Kutta method is called a p-th order method if the stated expansions coincide up to terms containing Δt_r^p. Since the relations thus obtained for determining coefficients $\gamma_i^{(p)}$, $\alpha_i^{(p)}$, $\beta_{ij}^{(p)}$ for any fixed order p yield, as a rule, a non-unique solution, this circumstance explains the great variety of Runge-Kutta equations of the same order.

As seen by the structure of equations (4.28) and (4.29), special procedures for finding the solution at several previous points are not required for obtaining the solution at point $t_r + \Delta t_r$, and only information about x_r is required.

For a second-order differential equation (4.25') the Runge-Kutta equations are analogous in form to (4.28) and (4.29):

$$\left.\begin{aligned} x_{r+1}&=x_r+\Delta t_r\dot{x}_r+\sum_{l=1}^{n_p}\bar{\gamma}_l^{(p)}k_l;\\ \dot{x}_{r+1}&=\dot{x}_r+\sum_{l=1}^{n_p}\bar{\bar{\gamma}}_l^{(p)}k_l, \end{aligned}\right| \qquad (4.30)$$

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where

$$k_1 = \Delta t_r \cdot f(t_r,\ x_r,\ \dot{x}_r);$$

$$k_l = \Delta t_r \cdot f\left(t_r + \bar{\alpha}_l \Delta t_r,\quad x_r + \bar{\bar{\alpha}}_l \Delta t_r \dot{x}_r + \sum_{j=1}^{l-1} k_j \bar{\bar{\beta}}_{lj},\right.$$

$$\left. \dot{x}_r + \sum_{j=1}^{l-1} k_j \bar{\beta}_{lj} \right) \quad (l = 2,\ 3,\ 4,\ \ldots,\ n_p).$$

Calculation equations. The equations given below are written for one scalar equation of system (4.24) or (4.25). In order to integrate the entire system (4.24) or (4.25) these equations must be applied to each component of vector x.

The equations for different variations of Runge-Kutta methods of the first, second, and third and fourth orders are presented in Tables 4.9 and 4.10 for numerical integration of first and second order differential equation systems.

In Runge-Kutta methods of orders $p \leqslant 4$ the number n_p of coefficients k_i in equation (4.28) and, consequently, the number of computations of the right sides $f(t, x)$ is equal to the order of the method: $n_p = p$, if $p \leqslant 4$. For $p = 5$, $n_p = 6$, and for $p > 7$, $n_p \geqslant p + 2$. High-order Runge-Kutta equations are quite complex and are not presented here.

When using the computer computations by the equations presented in Table 4.9 are programmed directly. Here it is advisable for solving complex problems to use methods for which are known simple expressions for local error (for example Runge-Kutta-Merson methods, Runge-Kutta-England methods). For methods with known local error programs of automatic step selection can be used (see below) which saves machine time for the same solution accuracy.

Accuracy of method. The exact expression for the summary error of approximate solution in numerical integration of differential equations by the Runge-Kutta methods is not known. A general apriori estimate (i.e., the estimate not related to the results of integration of a specific system) of the summary estimate for integration on segment $[t_0, t_r]$ of one first-order differential equation by the Runge-Kutta method is

$$|\tilde{x}_r - x(t_r)| \leqslant |\tilde{x}_0 - x(t_0)|\, e^{M(t_r - t_0)} + \left(L\Delta t^p + \frac{h}{\Delta t}\right) \frac{e^{M(t_r - t_0)} - 1}{M} \qquad /286$$

$$(r = 0,\ 1,\ 2,\ \ldots,\ N-1), \qquad (4.31)$$

where $x(t_r)$ is the exact solution of a differential equation at point t_r; $\tilde{x}_r$ is the solution found by the p-th order Runge-Kutta method;

p: $\tilde{x}_{r+1}=\tilde{x}_r+\Phi_f^{(p)}(t_r, \tilde{x}_r, \Delta t)+\delta_r$; δ_r is the round-off error; $\delta = \max \delta_r$ is the maximum round-off error; M, L are some constants, not depending on t, x, Δt, but depending on the form and properties of the right side $f(t, x)$; $\Phi_f^{(p)}$ is the general symbol for the p-th order Runge-Kutta equation.

The value $L\Delta t^p$ in equation (4.31) characterizes the effect of the local error of the method (error per step), and $\delta/\Delta t$ characterizes the effect of round-off error. It is obvious from the equation that an attempt to reduce local error leads to reduction of Δt, and consequently, to increase in round-off error. Moreover equation (4.31) shows that the estimate of summary error increases exponentially with integration interval $[t_0, t_r]$.

Practical methods of determining error of numerical integration on interval $[t_0, t_r]$. Theoretical formulas for apriori estimation of integration accuracy of the type (4.31) are rather rough and are based on information (constants L, M) not contained in the formula for $\Phi_f^{(p)}$ of the Runge-Kutta method. Therefore /287 they are used very little in practical computations. The practical method for determining the approximate aposteriori estimate of the error of numerical integration (i.e., of the estimate that uses information obtained during the calculation) is based on the Runge principle. The Runge principle consists in the use of asymptotic estimates (i.e., estimates valid for $\Delta t \to 0$) for evaluating the errors for finite Δt. Although asymptotic estimates do not give guaranteed limits of error they are, as a rule, very close to the limits and produce results that are satisfactory from the practical point of view. Furthermore asymptotic estimates can be used for evaluating the behavior of the actual error of the method.

Let $x(t_r)$ be the exact and x_r be the approximate solutions of the differential equation at point $t_r = r\Delta t + t_0$, found by a p-th order method with a constant integration step Δt (either the difference method or the Runge-Kutta method can be used). Then, for differential equations with sufficiently smooth right sides the expression for the error is

$$x_r - x(t_r)=\eta(t_r)\Delta t^p+O(\Delta t^{p+1}), \tag{4.32}$$

where $\eta(t_r)$ is some function depending on t_r and not on Δt.

To estimate the error it is necessary to integrate system (4.24) twice on segment $[t_0, t_r]$: once with step Δt and once with step $\Delta t/2$ (or $2\Delta t$, which is less reliable but more economical from the standpoint of computer time). Then it is possible to find on the basis of equation (4.32) the following estimate:

$$x_{2r}^{(\Delta t/2)} - x(t_r) \approx \frac{1}{2^p-1}(x_r^{(\Delta t)} - x_{2r}^{(\Delta t/2)}). \tag{4.33}$$

TABLE 4.9. RUNGE-KUTTA EQUATIONS OF DIFFERENT ORDERS FOR INTEGRATING FIRST-ORDER DIFFERENTIAL EQUATIONS $\dot{x} = f(t, x)$

General Name of the Method	Name of the Specific Method	Equations for Computing x_{r+1}	Equations for Computing Coefficients k_i
First-order Runge-Kutta method	Euler's method, broken method	$x_{r+1} = x_r + k_1$	$k_1 = \Delta t \cdot f(t_r, x_r)$
Second-order Runge-Kutta method	Improved Euler method	$x_{r+1} = x_r + \frac{1}{2}(k_1 + k_2)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f(t_r + \Delta t, x_r + k_1)$
	Euler's method with conversion (modified Euler method)	$x_{r+1} = x_r + k_2$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t, x_r + \frac{1}{2}k_1\right)$
Third-order Runge-Kutta method		$x_{r+1} = x_r + \frac{1}{6}(k_1 + 4k_2 + k_3)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t, x_r + \frac{1}{2}k_1\right)$ $k_3 = \Delta t \cdot f(t_r + \Delta t, x_r - k_1 + 2k_2)$
	Runge-Kutta-Haney method	$x_{r+1} = x_r + \frac{1}{4}(k_1 + 3k_3)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{3}\Delta t, x_r + \frac{1}{3}k_1\right)$ $k_3 = \Delta t \cdot f\left(t_r + \frac{2}{3}\Delta t, x_r + \frac{2}{3}k_2\right)$
		$x_{r+1} = x_r + \frac{1}{9}(2k_1 + 3k_2 + 4k_3)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t, x_r + \frac{1}{2}k_1\right)$ $k_3 = \Delta t \cdot f\left(t_r + \frac{3}{4}\Delta t, x_r + \frac{3}{4}k_2\right)$

TABLE 4.9. (CONTINUED)

General Name of the Method	Name of the Specific Method	Equations for Computing x_{r+1}	Equations for Computing Coefficients k_i
Fourth-order Runge-Kutta method	Standard Runge-Kutta method ("1/6" rule)	$x_{r+1} = x_r + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t,\ x_r + \frac{1}{2}k_1\right)$ $k_3 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t,\ x_r + \frac{1}{2}k_2\right)$ $k_4 = \Delta t \cdot f(t_r + \Delta t,\ x_r + k_3)$
		If (4.24) is a scalar linear equation, then the estimate of local error is $E = \frac{8}{15}\frac{\lvert k_3 - k_2\rvert^3}{(k_4 - k_2)^2}$ (Scraton's control term)	
	Hill's second method	$x_{r+1} = x_r + \frac{1}{6}(k_1+3k_2+3k_3+k_4)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t,\ x_r + \frac{1}{2}k_1\right)$ $k_3 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t,\ x_r - \frac{1}{2}k_1 + k_2\right)$ $k_4 = \Delta t \cdot f\left(t_r + \Delta t,\ x_r + \frac{1}{2}k_2 + \frac{1}{2}k_3\right)$
	"3/8" rule	$x_{r+1} = x_r + \frac{1}{8}(k_1+3k_2+3k_3+k_4)$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{3}\Delta t,\ x_r + \frac{1}{3}k_1\right)$ $k_3 = \Delta t \cdot f\left(t_r + \frac{2}{3}\Delta t,\ x_r - \frac{1}{3}k_1 + k_2\right)$ $k_4 = \Delta t \cdot f(t_r + \Delta t,\ x_r + k_1 - k_2 + k_3)$
	Hill's first method	$x_{r+1} = x_r + \frac{1}{6}\left[k_1 + 2\left(1 - \sqrt{\frac{1}{2}}\right)k_2 + 2\left(1 + \sqrt{\frac{1}{2}}\right)k_3 + k_4\right]$	$k_1 = \Delta t \cdot f(t_r, x_r)$ $k_2 = \Delta t \cdot f\left(t_r + \frac{1}{2}\Delta t,\ x_r + \frac{1}{2}k_1\right)$ $k_3 = \Delta t \cdot f\left[t_r + \frac{1}{2}\Delta t,\ x_r - \left(\frac{1}{2} - \sqrt{\frac{1}{2}}\right)k_1 + \left(1 - \sqrt{\frac{1}{2}}\right)k_2\right]$ $k_4 = \Delta t \cdot f\left[t_r + \Delta t,\ x_r - \sqrt{\frac{1}{2}}k_2 + \left(1 + \sqrt{\frac{1}{2}}\right)k_3\right]$

TABLE 4.9. (CONTINUED)

General Name of the Method	Name of the Specific Method	Equations for Computing x_{r+1}	Equations for Computing Coefficients k_i
Fourth-order Runge-Kutta method	Runge-Kutta-Merson method	$x_{r+1}=x_r+\frac{1}{2}(k_1+4k_4+k_5)$	$k_1=\frac{1}{3}\Delta t\cdot f(t_r,\ x_r)$ $k_2=\frac{1}{3}\Delta t\cdot f\left(t_r+\frac{1}{3}\Delta t,\ x_r+k_1\right)$ $k_3=\frac{1}{3}\Delta t\cdot f\left(t_r+\frac{1}{3}\Delta t,\ x_r+\frac{1}{2}k_1+\frac{1}{2}k_2\right)$ $k_4=\frac{1}{3}\Delta t\cdot f\left(t_r+\frac{1}{2}\Delta t,\ x_r+\frac{3}{8}k_1+\frac{9}{8}k_3\right)$ $k_5=\frac{1}{3}\Delta t\cdot f\left(t_r+\Delta t,\ x_r+\frac{3}{2}k_1+6k_4-\frac{9}{2}k_3\right)$ Estimate of local error (Merson's control term) $E=\frac{1}{5}\left(k_1-\frac{9}{2}k_3+4k_4-\frac{1}{2}k_5\right)=$ $=\begin{cases}O(\Delta t^5) & \text{if system (4.24) is linear or nearly linear}\\ O(\Delta t^4) & \text{if system (4.24) is nonlinear}\end{cases}$
	Runge-Kutta-England method	$x_{r+1}=x_r+\frac{1}{6}(k_1+4k_3+k_4)$	$k_1=\Delta t\cdot f(t_r,\ x_r)$ $k_2=\Delta t\cdot f\left(t_r+\frac{1}{2}\Delta t,\ x_r+\frac{1}{2}k_1\right)$ $k_3=\Delta t\cdot f\left[t_r+\frac{1}{2}\Delta t,\ x_r+\frac{1}{4}(k_1+k_2)\right]$ $k_4=\Delta t\cdot f(t_r+\Delta t,\ x_r-k_2+2k_3)$ $k_5=\Delta t\cdot f\left[t_r+\frac{18}{27}\Delta t,\ x_r+\frac{1}{27}(7k_1+10k_2+k_3)\right]$ $k_6=\Delta t\cdot f\left[t_r+\frac{1}{5}\Delta t, x_r+\frac{1}{625}(28k_1-125k_2+$ $+546k_3+54k_4-378k_5)]$ Estimate of local error of Runge-Kutta-England method: $E=\frac{1}{336}(-42k_1-224k_2-21k_4+162k_5+$ $+125k_6)=O(\Delta t^5)$

TABLE 4.10. FOUR POINT RUNGE-KUTTA EQUATIONS FOR SECOND ORDER DIFFERENTIAL EQUATIONS $\ddot{x} = \psi(t, x, \dot{x})$ (ORDER OF ACCURACY p = 4 ORDER $O(\Delta t^5)$)

Name of Method	Equations for Computing p. 286	Coefficients k_i
Standard fourth-order Runge-Kutta method for second order equations ("1/6" rule)	$x_{r+1}=x_r+\dot{x}_r\Delta t+$ $+\frac{1}{6}(k_1+k_2+k_3)\Delta t$ $\dot{x}_{r+1}=\dot{x}_r+\frac{1}{6}(k_1+$ $+2k_2+2k_3+k_4)$	$k_1=\Delta t\cdot\varphi(t_r,\ x_r,\ \dot{x}_r)$ $k_2=\Delta t\cdot\varphi\left(t_r+\frac{1}{2}\Delta t,\ x_r+\frac{1}{2}\dot{x}_r\Delta t+\right.$ $\left.+\frac{1}{4}k_1\Delta t,\ \dot{x}_r+\frac{1}{2}k_1\right)$ $k_3=\Delta t\cdot\varphi\left(t_r+\frac{1}{2}\Delta t,\ x_r+\right.$ $\left.+\frac{1}{2}\dot{x}_r\Delta t+\frac{1}{4}k_1\Delta t,\ \dot{x}_r+\frac{1}{2}k_2\right)$ $k_4=\Delta t\cdot\varphi(t_r+\Delta t,\ x_r+\dot{x}_r\Delta t+$ $+\frac{1}{2}k_2\Delta t,\ \dot{x}_r+k_3)$

Noteworthy in regard to formula (4.33) are the following: 1) round-off errors were not taken into account in derivation of the formula, therefore it is not applicable for calculations in which round-off errors prevail (for instance, for very large integration intervals and for differential equations with very complex right sides); 2) the formula is not valid for differential equations with discontinuous or piece-wise smooth right parts. In this case, however, the behavior of the actual error can be analyzed on the basis of formula (4.33) by integrating by intervals between points of discontinuity (the points of discontinuity themselves should be determined quite accurately, for example, by interpolation).

Local error and practical evaluation methods. Runge-Kutta methods of the p-th order exhibit local error (error per step of length Δt) equal to $O(\Delta t^{p+1})$. The main term, i.e., the dominating value for Δt, of this error is /288 $K_p\Delta t^{p+1}$. The order of the local error stated in Table 4.9 is valid only for differential equations whose right parts f(t, x) have continuous partial derivatives of t and x up to the order equal to the order of the method. In cases when continuity or continuous differentiability of right parts f(t, x) is violated at some point t^+ care must be taken to assure that the point t^+ does not fall within interval Δt. Otherwise it will be necessary to determine the point $(t^+, x(t^+))$ by interpolation, for instance, and then to perform computations from the point $(t^+, x(t^+))$ as the starting point.

One disadvantage of Runge-Kutta methods is that no exact expressions are known for local error in most of the methods, and in cases when they are known they are difficult to use. Thus, the exact formulas for the local error of first and second order Runge-Kutta methods are of the form

$$E=\frac{(\Delta t)^2}{2}\ddot{x}(\xi), \quad t_r\leqslant\xi\leqslant t_{r+1}, \tag{4.34}$$

(for the Euler method);

$$E=\frac{\theta}{4}(\Delta t)^3\dddot{x}(\xi), \quad t_r\leqslant\xi\leqslant t_{r+1}, \quad -1\leqslant\theta\leqslant 1 \tag{4.35}$$

(for the improved Euler method).

Expressions (4.34) and (4.35) show that the magnitude of local error of first and second order Runge-Kutta methods depends on derivatives of p + 1 order (p = 1, 2) of the desired solution $x(t)$ at some point ξ within segment of integration $[t_r, t_{r+1}]$. Hence follows the qualitative conclusion that when integrating differential equations whose solutions $x(t)$ have large $\ddot{x}(t), \dddot{x}(t)$ it is necessary to use smaller steps Δt than in the case of equations with small $\ddot{x}(t), \dddot{x}(t)$. Since the explicit expressions for $\ddot{x}, \dddot{x}$ are often extremely unwieldy formulas using only the computed values are applied instead of formulas of the type (4.34). (Such formulas usually yield an approximate estimate of local error with an accuracy of 10-20%.) Thus, for the Euler method the estimate of local error is

$$E=\frac{\Delta t}{2}[f_{r+1}-f_r]=\frac{1}{2}[k_1(t_{r+1}, x_{r+1})-k_1(t_r, x_r)]=O(\Delta t^2). \tag{4.34'}$$

In order to use formulas of type (4.34) it is necessary to perform one additional computation of the right side at point (t_{r+1}, x_{r+1}).

For the modified Euler method /289

$$x_{r+1}=x_r+k_2+O(\Delta t^3),$$

where $k_1=\Delta t f(t_r, x_r)=\Delta t f_r$; $k_2=\Delta t f(t_r+\Delta t/2, x_r+k_1/2=\Delta t f_{r+1/2}$, the estimate of local error is

$$E=\frac{\Delta t}{12}[5f_{r+1}-12f_{r+1/2}+8f_r-f_{r-1}]. \tag{4.35'}$$

Its use requires an additional computation of the right side at the point (t_{r+1}, x_{r+1}) and one previously computed f_{r-1}.

The best known of third-order Runge-Kutta methods is the one with which it is possible to increase successively the order of the method with the estimate of local error:

Of the first-order method

$$x_{r+1}^{(1)}=x_r+k_1, \quad k_1=\Delta t f(t_r, x_r).$$

Of the second-order method

$$x_{r+1}^{(2)}=x_r+\frac{1}{4}(k_1+3k_2), \quad k_2=\Delta t f\left(t_r+\frac{2}{3}\Delta t, x_r+\frac{2k_1}{3}\right).$$

The estimate of the error of the first-order method is

$$E^{(1)}=x_{r+1}^{(2)}-x_{r+1}^{(1)}=-\frac{3}{4}(k_1-k_2)=O(\Delta t^2).$$

The third-order method is

$$x_{r+1}^{(3)}=x_r+\frac{1}{6}[2k_1+3(k_2+k_3)], \quad k_3=\Delta t f\left(t_r+\frac{2\Delta t}{3}, \quad x_r+\frac{2k_2}{3}\right) \tag{4.36}$$

The estimate of the error of the second-order method is

$$E^{(2)}=x_{r+1}^{(3)}-x_{r+1}^{(2)}=-\frac{3}{8}(k_2-k_3)=O(\Delta t^3).$$

The local error of the standard fourth-order Runge-Kutta method can be found using the following expression:

$$E=3\Delta t f(t_{r+1}, x_{r+1})+k_1-2k_3-2k_4, \tag{4.37}$$

which requires computation of $x(t_{r+1}, x_{r+1})$.

Another method of estimating local error, often used in standard subprograms in cases when the explicit expression for error is unknown, consists in integration with two different steps. If integration results are obtained from point t_r to point t_{r+1} with steps Δt and $\Delta t/2$, then the error can be estimated using formula (4.33). This method, however, requires a great deal of computer time (the right part f(t, x) must be computed eleven times). Integration with steps Δt and $2\Delta t$ can be done to save computer time and the estimate can be found in one step. In this case three additional computations must be carried out for every eight computations of f(t, x) in order to obtain the estimate of local error. /290

Convenient estimates of local errors exist for the Runge-Kutta-Merson and Runge-Kutta-England methods (see Table 4.9). It should be pointed out that the estimate of local error of the Runge-Kutta-Merson method pertains, strictly speaking, to the case of integration of one linear equation. Nevertheless it is often used in practice and for integrating nonlinear equation systems. The same is true of Straton's control term for standard fourth-order Runge-Kutta method (see Table 4.9). The estimate of local error in the Runge-Kutta-England method is applicable in the case of integration of nonlinear equation systems, but it requires two additional computations of the right sides $f(t, x)$.

Choice of integration step. The following general guidelines are used for choosing the integration step.

1. The integration step should not be too large, since error increases as the step increases.

2. The integration step should not be too small, since round-off error increases as the step is reduced. Moreover, integration with a small step leads to large expenditures of computer time.

If at several characteristic points t_j of integration segment $[t_0, T]$, for m controlled vector components, the solutions $x_i(t)$, $i = 1, 2, \ldots, m \leqslant n$ of system (4.24) are known values $\hat{\varepsilon}_{ij}$, $i = 1, 2, \ldots, m$ of the permissible error of numerical solution, then the estimate of applicability of the selected constant step Δt can be arrived at on the basis of the Runge principle (see above). For this purpose system (4.24) is integrated twice: once with constant step Δt and once with constant step $2\Delta t$, and the integration error at selected point t_j is computed using formula (4.33). If the errors of controlled components thus computed do not exceed the given values integration step Δt is considered acceptable. Otherwise the size of the constant integration step should be decreased. We will note that this method of evaluating the size of the step does not take into account round-off errors.

In many flight mechanics problems (particularly in variational problems) integration with a constant step is not advantageous, since on some segments the step may be larger than the constant value which is permissible for the entire interval $[t_0, T]$. Integration with automatic step choice is employed for the purpose of cutting down computer operation while simultaneously maintaining the required numerical solution accuracy. Computer methods of automatic step choice are based on the use of estimates of local error and for integration of systems of the type (4.24) are constructed as follows. From n components of vector x are taken m components, $m \leqslant n$, the integration accuracy of which must be controlled. For the controlled x_i are set aside the values ε_i, $i = 1, 2, \ldots, m$ of permissible errors per integration step. Then, if during the process of integration from point (t_r, x_r) the current estimate of local error E_i for the i-th component exceeds the permissible value ε_i (even

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for one of the m controlled components), then the step Δt_{r+1} is decreased (usually cut in half) and the calculation is started over from the point (t_r, x_r). If the current value E_i satisfies the condition $E_i < \varepsilon_i/2^{p+1}$ for all $i = 1, 2, \ldots, m$, then the step is doubled and integration with this step is initiated from point (t_{r+1}, x_{r+1}). In other cases integration is done from the point (t_{r+1}, x_{r+1}) with the former step. (Doubling or halving the step involves the need to obtain either integral or fractional values for moments of time t_r, but with a small number of signs).

Thus the program for automatically changing the step during integration may be the following:

$$\Delta t_{r+1} = \begin{cases} \Delta t_r/2, & \text{if } E_i > \varepsilon_i \text{ for even one } i = 1, 2, \ldots, m; \\ \Delta t_r, & \text{if } \varepsilon_i/2^{p+1} < E_i < \varepsilon_i \text{ for all } i = 1, 2, \ldots, m; \\ 2\Delta t_r, & \text{if } E_i < \varepsilon_i/2^{p+1} \text{ for all } i = 1, 2, \ldots, m, \end{cases} \tag{4.38}$$

where Δt_{r+1} is the step of integration from the point (t_{r+1}, x_{r+1}), if integration must be done with the former or double step, and from point (t_r, x_r) if integration must be done with a reduced step.

In cases when the values of t_r need not be integral the formula for automatic step change can be:

$$\Delta t_{r+1} = \alpha \Delta t_r \min_{i=1, 2, \ldots, m} (\varepsilon_i/E_i)^{1/(p+1)}, \tag{4.39}$$

where $\alpha < 1$ (it may be assumed that $\alpha = 0.85$-0.95).

In cases when

$$\min_{i=1, 2, \ldots, m} (\varepsilon_i/E_i)^{1/(p+1)} > 1,$$

integration with step Δt_{r+1} is accomplished from point (t_{r+1}, x_{r+1}), and in cases when

$$\min_{i=1, 2, \ldots, m} (\varepsilon_i/E_i)^{1/(p+1)} < 1,$$

integration with step Δt_{r+1} is repeated from point (t_r, x_r).

The choice of initial step Δt_0 for system (4.38) is not very important, since the required step is found rather quickly, even when $T - t_0 = \Delta t_0$ is used as Δt_0. For system (4.39) the initial step Δt_0 should be made close to the actual value. Integration methods with automatic choice of step usually reduced by 20-30% the time required for finding the solution compared with integration methods with a constant step (assuming identical accuracy). Here the second method of automatic step change is somewhat more effective than the first. /292

Features of application of Runge-Kutta methods using digital computer. Algorithms of integration of differential equation systems by various Runge-Kutta methods are included in the libraries of standard subprograms of nearly all Soviet digital computers.

The choice of algorithm depends both on the type of computer and complexity of the problem to be solved. Fourth-order Runge-Kutta methods are most commonly used for flight mechanics problems. The rather high accuracy of fourth-order methods enables integration with a step that is not too small and for many problems ensures acceptable computer time. When solving some flight mechanics problems it becomes necessary to place the program of the entire problem in the operational memory of the computer. In such cases all algorithms should have not only high speed, but a short program. Since the Runge-Kutta methods (in contrast to Adams' and the forecast-correction methods, cf. Sections 4.1.1 and 4.1.2), do not require separate procedures for finding auxiliary initial (starting) points, their programs are shorter than those of difference methods of the same order. Certain modifications of the Runge-Kutta methods accord additional saving of computer memory. Thus the Runge-Kutta-Hill method (see Table 4.9) can be written in a form that requires only 3n groups of memory cells (instead of 4n groups required by the standard fourth-order Runge-Kutta method) for solution with a digital computer. Moreover, when the method is written in such form all coefficients k_i, i = 1, 2, 3, 4 are computed in the same cycle of the program. The stated form of the Runge-Kutta-Hill method (equivalent to Hill's first method, cf. Table 4.9) is:

$$\left.\begin{aligned} x_{r+1}^{(1)} &= x_r + 0.5(k_1 + q_1); \\ x_{r+1}^{(2)} &= x_{r+1}^{(1)} + (1 - \sqrt{0.5})(k_2 + q_2); \\ x_{r+1}^{(3)} &= x_{r+1}^{(2)} + (1 + \sqrt{0.5})(k_3 + q_3); \\ x_{r+1} &= x_{r+1}^{(4)} = x_{r+1}^{(3)} + \frac{1}{6}(k_4 + q_4), \end{aligned}\right\} \qquad (4.40)$$

where

$$q_1 = 0, \quad q_2 = -(k_1 + aq_1);$$

$$q_3 = -(2 - \sqrt{2})\left(k_2 + \frac{2 - 3\sqrt{0.5}}{2 - \sqrt{2}} q_2\right);$$

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$$q_4 = -2(2+\sqrt{2})\left(k_3 + \frac{2+3\sqrt{0.5}}{2+\sqrt{2}} q_3\right);$$

$$k_1 = \Delta t f(t_r, x_r), \quad k_2 = \Delta t f(t_r + \Delta t/2, x_{r+1}^{(1)});$$

$$k_3 = \Delta t f(t_r + \Delta t/2, x_{r+1}^{(2)});$$

$$k_4 = \Delta t f(t_r + \Delta t, x_{r+1}^{(3)}).$$

The numerical value of coefficient a in this system is not important, since it is multiplied by zero.

Many standard programs for integrating differential equation systems of the form (4.24) call for substitution of the independent variable t in the right side of these equations by a new dependent variable x_0. The substitution is done by attaching to system (4.24) the differential equation

$$\frac{dx_0}{dt} = 1 \qquad (4.41)$$

with initial condition

$$x_0(t_0) = t_0.$$

The inclusion of auxiliary equation (4.41) in general system (4.24) permits automatic computation of the zero values of the independent variable t (for instance, t_r, $t_r^* + \Delta t/2$, $t_r + \Delta t$, etc.) by integration, and not by adding the corresponding step fractions. Moreover the introduction of new variable x_0 facilitates formulation of the last integration step in cases when time T of completion of the integration process is determined by the condition of coincidence of some dependent variable $x_i(t)$ and apriori given x_i^*. In these cases the last step is determined by dividing the difference $x_i - x_i^*$ by the known derivative $x_i(t_0, x)$.

Comparative advantages and disadvantages of Runge-Kutta methods.

1. The Runge-Kutta methods are quite simple and have a standard computational structure; programs for computers that execute these methods are quite compact.

2. When integrating differential equations by the Runge-Kutta methods it is not necessary to compute extra starting points, as in the case of Adams and forecast-correction methods. All the information required for finding the solution in a given step is obtained by computing the right sides of the equations at several special auxiliary points. /294

3. When a method is available for computing the local error in Runge-Kutta methods programs for automatic step change during integration process are readily executed. Here, in contrast to the Adams and forecasting-correction methods, changing the step does not require additional computations (for instance of new starting points or interpolation) or storing of the results of previous steps.

4. During integration of differential equations with discontinuous or insufficiently smooth right parts, in order to maintain the order of the method it is necessary to reinitiate integration with each passage through the point of discontinuity or violation of smoothness (this point is found by interpolation and is used as the new starting point). The Runge-Kutta methods are advantageous for integration of such equations in that extra computations are not required for conversion to new initial conditions. Sometimes the stated equations are integrated with a small step without determination of points of discontinuity. In such cases the presence of a point of discontinuity in the Runge-Kutta methods has a direct effect only on one step, whereas in the Adams and forecasting-correction methods several subsequent steps are affected, depending on the number of starting points.

5. The Runge-Kutta methods are characterized by multiple access to computation of the right sides of differential equations in one step. Thus, for methods of order $p \leqslant 4$ p-multiple computation of the right part $f(t, x)$ of equation (4.24) is required. In the case of complex right parts this involves a great deal of computer operation. The Adams or forecasting-correction methods of any order require only 1-3-multiple computation of the right side.

6. Determination of the local error in the Runge-Kutta methods requires extra computations, whereas in the forecasting and correction methods the local error is determined as a by-product of the computations.

7. If the effect of round-off errors is ignored, the accuracy of integration by the Runge-Kutta methods can be increased by decreasing step Δt and by converting to a higher order. The Runge-Kutta methods, as a rule, do not permit conversion to equations of higher order in one computing step (method (4.36) is an exception), and therefore a change of order is accompanied by a change of program. Furthermore high-order Runge-Kutta methods require a large number of computations of the right sides ($p + 2$ computations for methods of order $p \geqslant 6$). At the same time the difference equations of Adams' method /296
permit conversion to higher order methods practically without changing the computing operation; in this case the conversion is accompanied by an increase in the number of computations of the right sides of the equations, but the number of starting points should increase correspondingly.

8. The Runge-Kutta methods are computationally stable, but the range of stability is small. Therefore, when integrating equations of motion with very different time constants the permissible step Δt may be limited by its stability conditions rather than by the local error of the method.

TABLE 4.11. RESULTS OF INTEGRATION OF EQUATIONS OF MOTION BY VARIOUS RUNGE-KUTTA METHODS IN HIGH SPEED DIGITAL COMPUTER WITH STEP Δt = 1 sec

t in sec	Parameters of Motion	Parameters of Motion Computed by Method: First-order Runge-Kutta (Euler method)	Second order Runge-Kutta (Euler with conversion method)	Third-order Runge-Kutta (Runge-Kutta-Haney method)	Fourth-order Runge-Kutta (Standard Runge-Kutta method)
0	v, m/sec	500,00000	500,00000	500,00000	500,00000
	θ, deg	8,0000000	8,0000000	8,0000000	8,0000000
	h, m	12000,000	12000,000	12000,000	12000,000
1	v	500,49950	500,40354	500,40169	500,40232
	θ	8,9099438	8,9107658	8,9108502	8,9108370
	h	12069,586	12073,552	12073,548	12073,550
2	v	500,80722	500,60813	500,60440	500,60564
	θ	9,8216009	9,8237420	9,8239148	9,8238896
	h	12147,104	12155,019	12155,011	12155,014
3	v	500,91630	500,60676	500,60113	500,60297
	θ	10,735602	10,739585	10,739850	10,739814
	h	12232,536	12244,374	12244,362	12244,367
4	v	500,81986	500,39246	500,38490	500,38731
	θ	11,652598	11,658971	11,659334	11,659289
	h	12325,842	12341,584	12341,569	12341,576
5	v	500,51102	499,95823	499,94869	499,95168
	θ	12,573257	12,582601	12,583066	12,583013
	h	12426,996	12446,610	12446,592	12446,600
10	v	495,53968	494,24294	494,22319	494,22870
	θ	17,256615	17,290855	17,291923	17,291856
	h	13048,610	13086,956	13086,918	13086,941
20	v	465,65994	462,21093	462,16787	462,17605
	θ	27,376625	27,535936	27,538942	27,539092
	h	14833,866	14903,610	14903,497	14903,574
40	v	284,03406	267,08954	266,89992	266,82796
	θ	57,621190	59,527634	59,551980	59,569621
	h	19763,512	19793,989	19792,920	19792,634

Note: Commas indicate decimals

Example 5. Numerical integration of equations of motion by Runge-Kutta methods of different orders. /296

Explained for the problem described in example 3, Section 4.1.1, is integration of the equations of motion of an aircraft by Runge-Kutta methods of the first, second, third and fourth orders with constant step $\Delta t = 1.0$ sec. The calculations were done in a high speed digital computer and the programs were written for computing the right sides of the equations of motion and for computations by the equations of various Runge-Kutta methods. The results of the calculations are summarized in Table 4.11, where they are given to the eight significant figures printed out by the computer. The more exact names of the methods corresponding to Table 4.9 are indicated in the headings of this table.

4.1.4. General Comments and Recommendations on the Choice and Application of Methods of Numerical Integration in Flight Mechanics Problems

1. For simple problems, placing a small load on the computer memory bank, it is advisable to use existing standard numerical integration subprograms.

2. In complex problems, placing a large load on the computer memory bank, and also when using a digital computer with a small operational memory it is better (for economy of memory) to program independently one of the above described methods (depending on the specific problem and requirements on the accuracy of the solution).

3. When memory volume is limited and the differential equations have sufficiently simple right parts it is advisable to use Runge-Kutta methods with a constant step.

4. If integration time must be reduced (for example when solving boundary problems by methods of reduction to a problem with initial data, when it is necessary in each iteration to integrate repeatedly the initial differential equation system) it is advisable to use the forecasting and correction methods with a constant or variable step or Runge-Kutta methods with a variable step.

5. Experience in the solution of flight mechanics problems indicates that lower order methods (Euler's method, improved Euler's method) with a constant step, when solving complex problems of large dimensionality, often give substantial error (when step Δt is large the error of the method is large, and when step Δt is small the round-off error is large) or require considerable computer time. The use of second-order methods with automatic step selection (Runge- /297
-Kutta methods or forecasting-correction method) for solving simple trajectory problems produces quite satisfactory results.

6. In the absence of preliminary data about the integration step in some specific problem numerical experiment is required before mass or final calculations can be done. The experiment usually employs some standard subprogram (SS) of numerical integration with automatic step selection (for instance, the SS that executes the standard fourth-order Runge-Kutta method with double calculation with steps Δt and $2\Delta t$).

7. The Adams and forecast-correction methods are recommended for integration with a constant step (particularly if the integration step is known ahead of time). The "starting" points for this purpose should be found by the fourth-order Runge-Kutta methods, for instance by the Runge-Kutta-Hill method.

8. In problems with a large number of computations (for instance when integrating on large segments of time) measures should be taken to minimize the effect of round-off errors. Statistical analysis of round-off errors during integration of the equations of motion of celestial mechanics indicates that under certain assumptions the round-off errors increase approximately in proportion to $n^{3/2}$, where n is the total number of integration steps. This means that the random round-off error in the last bit is converted after 1,000 steps to the third low-order digit. The estimate of the number k of lost decimal bits is sometimes determined by the formula $k = \log_{10} 0.1124 n^{3/2}$, also established for celestial mechanics problems. The number of lost bits, computed according to this formula, is given in Table 4.12.

TABLE 4.12

n	7	31	144	670	3100	14400
k	1	2	3	4	5	6

The actual number of lost bits in this case is smaller than the number stated in the table with the probability of 99.7%, i.e., the data in the table give the top estimate.

9. To maintain the order of method (order of local error) during integration of differential equations with discontinuous or insufficiently smooth right parts, the calculation must be started over from the point of discontinuity (or point of violation of smoothness) as the initial point. In the case of many discontinuities the Runge-Kutta methods are quite appropriate, since they do not require determination of "starting" points. Points of discontinuity of the right sides or of their derivatives should be determined quite accurately, for example by using interpolation equations of the corresponding order. /298

4.2. Numerical Methods of Calculating Equilibrium Flight Modes and Methods of Solving Boundary Problems of Flight Mechanics (Numerical Methods of Solving Nonlinear Equation Systems)

Sufficiently complete equations of motion of aircraft consist of differential equations of motion of the center of mass, differential equations of motion of the aircraft relative to the center of mass, kinematic differential and algebraic equations and differential equations of change of mass and of moments of inertia of the aircraft.

Such an equation system, generally speaking, describes unsteady motion of an aircraft and its analysis involves the use of methods of numerical integration of differential equations of motion (see Section 4.1). Before complex unsteady motion of an aircraft can be analyzed it is often advisable to examine certain important special cases of motion. One such case is the equilibrium flight mode, in which the sum of acting external forces and moments are identically equal to zero.

The flight characteristics of aircraft in equilibrium flight modes are described by systems of nonlinear algebraic or transcendental equations.

In the general case equations of motion are

$$\dot{x}_i = f_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) \quad (i = 1, 2, \ldots, n); \tag{4.42}$$

$$\dot{z}_j = \varphi_j(\mathbf{x}, \mathbf{z}) \quad (j = 1, 2, \ldots, n). \tag{4.43}$$

Here $\mathrm{x} = (x_1, x_2, \ldots, x_n)^T$ is the vector of linear and (or) angular velocities; $\mathrm{z} = (z_1, z_2, \ldots, z_n)^T$ is the vector of linear and (or) angular locations; f_i are the projections of forces and moments; ϕ_j describe the kinematic relations; $\mathrm{u} = (u_1, \ldots, u_m)^T$ is the control vector.

Equilibrium flight modes are described by the conditions

$$f_i(\mathbf{x}, \mathbf{z}, \mathbf{u}) = 0 \quad (i = 1, 2, \ldots, n), \tag{4.44}$$

whence follows that in such modes $\mathrm{x} = \mathrm{x}_0 = \text{const}$.

Relations (4.44), for various fixed components of vector of state u, comprise a system of n nonlinear transcendental equations with n unknowns. The solutions of this system determine, in particular, the range of kinematic parameters, in which the equilibrium flight mode is possible.

Example 6. The equations of motion of the center of mass of an aircraft, completing flight without roll and skid in the vertical plane above the flat, stationary surface of the earth, being described in Cartesian vertical-velocity coordinate system $Ox_V y_V z_V$, are /299

$$m\dot{v} = P\cos(\alpha + \varphi_p) - Q(v, h, \alpha) - mg(h)\sin\theta; \tag{I}$$

$$mv\dot{\theta} = P\sin(\alpha + \varphi_p) + Y(v, h, \alpha) - mg(h)\cos\theta; \tag{II}$$

$$\dot{\xi} = v\cos\theta; \tag{III}$$

$$\dot{\eta} = \dot{h} = v\sin\theta; \tag{IV}$$

$$\dot{m} = \frac{P\,C_e(v, h, P)}{3600\,g}. \tag{V}$$

Here v is flight velocity; θ is the angle of inclination of the trajectory; ξ is longitudinal range; $\eta \equiv h$ is flight altitude; m is the mass of the aircraft; α is the angle of attack; ϕ_p is the angle of placement of the engine; P is thrust; Q is aerodynamic drag; Y is lift; g is gravitational acceleration.

The equilibrium mode is determined by the relations

$$P\cos(\alpha + \varphi_p) - Q(v, h, \alpha) - mg(h)\sin\theta \equiv 0; \tag{VI}$$

$$P\sin(\alpha + \varphi_p) + Y(v, h, \alpha) - mg(h)\cos\theta \equiv 0. \tag{VII}$$

whence follows that $v = v_0 = \text{const}$, $\theta = \theta_0 = \text{const}$.

If ϕ_p, m, h, v_0, θ_0 are known, then relations (VI) and (VII) represent a system of transcendental equations relative to P and α. The solutions P* and α* of system (VI) and (VII) determine the thrust and angle of attack of the equilibrium flight mode. Using these values it is possible to determine the mass flow rate per second $\dot{m}$ in the equilibrium flight mode.

Calculation of equilibrium modes is not the only example when it is necessary in flight mechanics to deal with the solution of systems of nonlinear algebraic or transcendental equations. There are many problems whose solutions amount in the final analysis to the solution of nonlinear equation systems. Boundary problems of flight mechanics, encountered in the analysis of optimal trajectories, are a common example (see Chapter 3). Let the equations of motion of aircraft be written as

$$x_i = f_i(t, \mathrm{x}) \quad (i = 1, 2, \ldots, n), \tag{4.45}$$

let for $t = t_0$ be given the initial conditions

$$x_i(t_0) = x_{i0} \quad (i = 1, 2, \ldots, k < n), \tag{4.46}$$

and for $t = t_1$ be given the final conditions

$$x_j(t_1)=x_{j1}^* \quad (j=k+1,\ k+2,\ \ldots,\ n), \tag{4.47}$$

or

$$x_l(t_1)=x_{l1}^* \quad (l=1,\ 2,\ \ldots,\ n-k). \tag{4.48}$$

The solution of boundary problem (4.45)-(4.47) or (4.45), (4.46), (4.48) cannot be found with the aid of the known numerical integration methods (see Section 4.1), since the latter are designed for the solution of a problem with completely known initial data for all components of the vector $x = (x_1, \ldots, x_n)^T$, i.e., for the so-called Cauchy problem. The problem at hand, however, can be reduced to the Cauchy problem if the deficient initial conditions are determined ahead of time for the components x_j, $j = k + 1, \ldots, n$ of vector x. For this purpose we assume /300

$$x_j(t_0)=\hat{x}_j \quad (j=k+1,\ k+2,\ \ldots,\ n), \tag{4.49}$$

where $\hat{x}_j$ are some unknown variables. If system (4.45) with initial conditions

$$x_l(t_0)=x_{l0} \quad (l=1,\ 2,\ \ldots,\ k);$$
$$x_j(t_0)=\hat{x}_j \quad (j=k+1,\ k+2,\ \ldots,\ n),$$

is integrated, then the components of vectors x_{j1}, $j = k + 1, \ldots, n$ (4.47) or x_{i1}, $i = 1, 2, \ldots, n - k$, at time t_1 of completion of integration, will be functions of the variables $\hat{x}_j$ (for fixed x_{i0}):

$$x_{j1}=F_j(\hat{x}_{k+1},\ \hat{x}_{k+2},\ \ldots,\ \hat{x}_n) \quad (j=k+1,\ k+2,\ \ldots,\ n);$$
$$x_{l1}=F_l(\hat{x}_{k+1},\ \hat{x}_{k+2},\ \ldots,\ \hat{x}_n) \quad (l=1,\ 2,\ \ldots,\ n-k).$$

In each of the stated cases, in order to satisfy given boundary conditions (4.47) or (4.48), it is necessary to solve a nonlinear system (n - k) of transcendental equations (n - k) with unknown $\hat{x}_j$:

$$x_{j1}-x_{j1}^*=F_j(\hat{x}_{k+1},\ \ldots,\ \hat{x}_n)-x_{j1}^*=0 \quad (j=k+1,\ k+2,\ \ldots,\ n). \tag{4.50}$$

or

$$x_{i1}-x_{i1}^{*}=\tilde{F}_i(\hat{x}_{k+1},\ldots,\hat{x}_n)-x_{i1}^{*}=0 \quad (i=1,2,\ldots,n-k). \tag{4.51}$$

Boundary problems are often encountered in problems of external ballistics and navigation, in problems of guidance and in interplanetary transit theory. Presented in this section are the fundamental definitions and concepts required for using methods of numerical solution of nonlinear (algebraic or transcendental) equation systems, and the methods themselves.

4.2.1. Iteration Methods of Solving Nonlinear Equation Systems

Equation system. The problem of calculating equilibrium and steady state flight modes and the methods of solving boundary value problems of flight dynamics, based on their reduction to a problem with initial conditions, amount to analysis of nonlinear algebraic or transcendental equation systems of the form /301

$$f_i(x_1, x_2, x_3,\ldots, x_n)=0 \quad (i=1, 2, 3,\ldots, n), \tag{4.52}$$

or in vectorial notation

$$\mathbf{f}(\mathbf{x})=0, \tag{4.53}$$

where

$$\mathbf{f}=\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}-$$

are n-dimensional column vectors.

Equation systems may differ in form, for instance:

$$g_i(x_1, x_2,\ldots, x_n)=h_i(x_1, x_2,\ldots, x_n) \quad (i=1, 2, 3,\ldots, n). \tag{4.54}$$

Nevertheless any equation system (4.54) is equivalent to a system of the form (4.52).

Solution of equation system. If in a general domain $X \subset R^n$ of n functions $f_i(x)$ with n independent variables $x_1, x_2, \ldots, x_n$ can be found a certain system of independent variables

$$\mathbf{x}^{(1)}=(x_1^{(1)}, x_2^{(1)},\ldots, x_n^{(1)}),\ \mathbf{x}^{(2)}=(x_1^{(2)}, x_2^{(2)},\ldots, x_n^{(2)}),\ldots,$$
$$\mathbf{x}^{(k)}=(x_1^{(k)}, x_2^{(k)},\ldots, x_n^{(k)}), \tag{4.55}$$

such that only for these values are the equations

$$\left.\begin{array}{c} f_1(x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}) = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}) = 0 \\ (j = 1, 2, 3, \ldots, k), \end{array}\right\} \tag{4.56}$$

valid, then each $x^{(j)}$ is called *a solution of equation system* (4.52); if, however, equations (4.56) are valid for all x, then they are called *identities*.

Possible solutions. An equation system of form (4.52) may have one solution, several, infinite solutions or no solution at all. Formally the problem of thorough analysis of a system involves determination of all possible solutions. From a practical standpoint, however, not all solutions are of the same importance. Thus, certain solutions may lie beyond the domain $D \subset X \subset R^n$ of vector x that is permissible in the terms of the problem. Moreover, both real and complex solutions of (4.52) can be examined. For flight dynamics problems /302 involving the calculation of equilibrium flight modes and solution of boundary value problems, the real solutions are of greatest importance; they will be discussed in ensuing sections. As concerns the problem of finding all solutions in given region D (region of "technical" solutions), however, the algorithms of solution of such problem are not yet well enough developed. Examined below are the methods of finding the solution that is closest in some sense to the selected initial approximation.

The number of solutions of an independent combined equation system of general form may vary in extremely wide limits. Thus, in the case when all f_i are polynomials of power m_i relative to $(x_1, x_2, \ldots, x_n)$, system (4.52) can have $N = \prod_{i=1}^{n} m_i$ solutions (real and complex).

Combined system. Equation system (4.52) is called combined if there exists for it components of vector x satisfying all equations of this system.

Independent system. Equation system (4.52) is called independent if the determinant of the Jacobi matrix J = df/dx does not vanish identically:

$$\det J = \det\left[\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}\right] = \det\begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots, & \frac{\partial f_1}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, & \ldots, & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \not\equiv 0. \tag{4.57}$$

Annihilation of the determinant of the Jacobi matrix usually complicates the computational process of finding the solutions.

Sub-definite system. If one or several equations of system (4.52) are corollaries of others, then the system has an infinite set of solutions. Such a system can be regarded as sub-definite: it has not more than (n - 1) equations with n unknowns.

Equivalent systems. Two systems of n equations with n unknowns:

$$f(x) = 0 \tag{4.58}$$

$$g(x) = 0 \tag{4.59}$$

with the same vector of independent variables x, examined in the same domain, are called equivalent if each solution of one is the solution of the other and conversely.

Isolated, simple and multiple solutions. If a system is combined and independent its solutions are most often isolated, i.e., in a sufficiently small vicinity of each solution $x^{(j)}$ are no other solutions. If, in addition, at the point of solution $x^{(j)}$ the determinant of the Jacobi matrix (4.57) is different from zero, the solution is called simple; otherwise the solution is multiple. /303

Computable functions. The fact that functions $f_i(x)$ are often given not explicitly, but through some more or less complex computational process, is a feature of applied flight dynamics problems. For instance, in solving boundary problems by the method based on reduction to a problem with initial conditions, functions $f_i(x)$ are the result of integration on segment $[t_0, t_1]$ of some differential equation system. In the calculation of equilibrium modes the aerodynamic forces and the force of gravity may be given by equations which include iteration processes, etc. In such cases it is said that the functions $f_i(x)$ are computable functions. Computable functions are defined as those which can be computed with a given accuracy for any independent variables in their domain, but whose explicit form is unknown. Some of its general properties (continuity, differentiability, etc.) in particular, may be unknowns.

All operations on such functions can be accomplished only by computing the values of the function itself. Differentiation, for example, cannot be done directly, but the partial derivatives of such a function can be computed at any point with the appropriate equations of numerical differentiation. These circumstances impose specific requirements on methods used for solving nonlinear equation systems with computable functions. First, these methods should not be based on knowledge of the explicit form of functions $f_i(x)$. Second, it is desirable that high-order derivatives not be required in them, since their

computation requires a great deal of computer time. Third, it is necessary to estimate the computational effectiveness of the numerical methods, which would take into account the computer resources required for computing $f_i(x)$ with comparatively simple operations in the computing process.

Finally, we should point out that in the case of computable vector-functions $f(x)$ situations may arise when the computation of some component of the vector $f(x)$ is automatically accompanied by computation of all other components of this vector. Thus, for instance, if a system of finite equations results from the solution of the boundary value problem by the method of reduction to a problem with initial data, then if some coordinate has to be found on the right end it becomes necessary to compute simultaneously all the other coordinates. This feature should be taken into consideration when choosing the numerical method. As a rule nonlinear equation systems which permit independent computation of the /304
components of vector $f(x)$ make it possible to organize more economical computer processes than systems in which all components of vector $f(x)$ must be computed simultaneously.

Idea of iterative process. Iterative and recurrent forms. Iteration methods (methods of successive approximations) for finding solutions are used for nonlinear equations of general form (4.52). The essence of these methods consists in special construction of a sequence of vectors $\{x^{(r)}\}$ that converges on the solution of equation system (4.52).

The series of vectors

$$\{x^{(r)}\}=\{x^{(0)}, x^{(1)}, \ldots, x^{(r)}, \ldots\}, \quad x^{(r)}=(x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}) \tag{4.60}$$

is called *iterative* if it is formed by computations according to the equation

$$x^{(r)}=\varphi_f(x^{(r-1)}) \quad (r=1, 2, 3, \ldots) \tag{4.61}$$

(the index "f" of the function ϕ_f indicates possible dependence on equation system $f(x) = 0$. In coordinate form the iterative sequence is

$$x_i^{(r)}=\varphi_{fi}(x_1^{(r-1)}, x_2^{(r-1)}, \ldots, x_n^{(r-1)}) \quad (r=1, 2, 3, \ldots). \tag{4.62}$$

The resulting sequence is called *iterative* in terms of the *coordinate set*.

By assigning in some way the initial approximation $x^{(0)}$ using expression (4.61), $x^{(1)}$, $x^{(2)}$ etc. can be computed in succession. Such uniformity of conversion from one term of the sequence to the next is extremely convenient for computations using general purpose digital computers.

If the components of $\phi_{fi}(x)$ can be computed independently, then the iterative sequence of vectors $\{x^{(r)}\}$ can be constructed by the "iterations by individual coordinates" plan (Seidel's method):

$$x_i^{(r)} = \varphi_{fi}(x_1^{(r)}, x_2^{(r)}, \ldots, x_{i-1}^{(r)}, x_i^{(r-1)}, x_{i+1}^{(r-1)}, \ldots, x_n^{(r-1)})$$
$$(i = 1, 2, \ldots, n; \quad r = 1, 2, 3 \ldots), \tag{4.63}$$

i.e., the components $x_1^{(r)}$, $x_2^{(r)}$, ..., $x_{i-1}^{(r)}$ which have already been found are used for determining the i-th component of vector x.

The use of recurrent sequence (iterative sequence "with memory") is a more general approach. Series of vectors $\{x^{(r)}\}$ is called recurrent (given by recurrent formula) if several ($p \geqslant 1$) of its terms $x^{(0)}$, $x^{(1)}$, ..., $x^{(p)}$ are given and the function by means of which $x^{(r)}$ is expressed through the preceding terms of the sequence is known: /305

$$x^{(r)} = \varphi_f(x^{(r-1)}, x^{(r-2)}, \ldots, x^{(r-p)}, x^{(r-p+1)})$$
$$(p \geqslant 1, \quad r = p+1, p+2, \ldots). \tag{4.64}$$

The methods of constructing specific functions ϕ_f, both for iterative and for recurrent sequence, are examined in the ensuing sections.

Convergence of sequence. Sequence of vectors $\{x^{(r)}\}$ is called convergent on vector x* if for each component of vectors $x^{(r)}$ and x* the relation

$$\lim_{r \to \infty} x_i^{(r)} = x_i^* \quad (i = 1, 2, 3, \ldots, n), \tag{4.65}$$

is valid.

In vectorial notation this condition is

$$\lim_{r \to \infty} x^{(r)} = x^*. \tag{4.66}$$

If the spacing (metric) $\rho(x^{(r)}, x^*)$, or the norm $\|x\|$ of vector x is introduced between points $x^{(r)}$ and x^* (for example, $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ is the Euclidean norm), the definition of convergence (4.66) can be written in equivalent form:

$$x^* = \lim_{r\to\infty} x^{(r)},$$

if

$$\lim_{r\to\infty} \varrho(x^{(r)}, x^*) = \lim_{r\to\infty} \|x^{(r)} - x^*\| = 0. \tag{4.67}$$

Fundamental sequence. The sequence $\{x^{(r)}\}$ is called fundamental (Cauchy sequence, or self convergent) if for each $\varepsilon > 0$ there exists an R such that

$$\|x^{(r)} - x^{(\bar{r})}\| < \varepsilon,$$

as soon as $r > R$ and $\bar{r} > R$ (otherwise r and $\bar{r}$ are any values).

Conditions of convergence. In order that sequence $\{x^{(r)}\}$ have a limit (be convergent) it is necessary that it be fundamental. In the general case the fundamentality of a sequence is no guarantee of convergence of the sequence. If, however, the metric or normed space is complete, fundamentality of a sequence is a sufficient indication of its convergence. The practical importance of fundamental sequences is that this property is readily checked during solution of the equation system ($x^{(r)}$ and $x^{(r-1)}$ are known), whereas convergence according to the norm $\|x^{(r)} - x^*\|$ cannot be established directly because x^* is unknown. /306

Another (indirect) indication of convergence of sequence $\{x^{(r)}\}$ to x^* is convergence of the sequence of the norm of the vector of discrepancies[18]

$$\|f^{(r)} - f^*\| = \|f(x^{(r)}) - f(x^*)\| = \|f(x^{(r)})\| \tag{4.68}$$

or fundamentality of sequence $\{f(x^{(r)})\}$, i.e., validity of the condition

$$\|f(x^{(r)}) - f(x^{(\bar{r})})\| < \varepsilon \tag{4.69}$$

for

$$r > R, \quad \bar{r} > R.$$

Order (rate) of convergence. The characteristics of rate of convergence of sequence $\{x^{(r)}\}$ (or $\{f(x^{(r)})\}$ are important from the standpoint of computations. Let $\{x^{(r)}\}$ be a sequence of vectors in X, which converges on element x^* of X. If there exist a positive real number v and non-zero constant C such that

[18] The vector of discrepancies is defined as the difference of vectors $f^{(r)} - f^*$.

$$\lim_{r\to\infty}\frac{\|x^{(r+1)}-x^*\|}{\|x^{(r)}-x^*\|^{\nu}}=C\ ^{19},\tag{4.70}$$

then number v is called the order of convergence (asymptotic rate of convergence) of sequence $\{x^{(r)}\}$ and the constant C is the asymptotic constant of convergence (in the case of recurrent sequences $v > 1$ may be a fraction).

Thus, if $v = 1$ the sequence has a first-order asymptotic convergence (linear), if $v = 2$ it has second-order (quadratic) convergence; if $1 < v < 2$ the sequence is said to have superlinear convergence.

It is sometimes possible, instead of (4.70), to find estimates of the form

$$\|x^{(r+1)}-x^*\|<q\|x^{(r)}-x^*\|,\quad q<1,\tag{4.71}$$

or

$$\|x^{(r+1)}-x^*\|<C\|x^{(r)}-x^*\|^2.\tag{4.72}$$

In the first case sequence $\{x^{(r)}\}$ is said to converge at a rate not less than linear (or not slower than the geometric progression with the denominator $q < 1$), and in the second case it is said to converge at a rate not less than quadratic. The advantage of quadratic convergence is, roughly speaking, that /307
$C < 1$ and $x^{(r)}$ produces k correct decimal points[20] of the solution, then the approximation of $x^{(r+1)}$ yields at least 2k correct decimal points (for method of order $v > 1$, generally speaking, $\Delta k_{i+1} = v\Delta k_i$, where Δk_{i+1} is the increment of the number of correct digits). In the case of linear convergence, however, the number of correct digits is determined by the relation $n_i = n_0 - i \log q$, i.e., changes considerably more slowly. Linear convergence is considered "fast" if $q < 1/10$, and "slow" if $1/2 < q < 1$.

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[19]Sometimes the order of convergence is defined as the limit $v=\lim_{r\to\infty}\frac{\log\|x^{(r+1)}-x^*\|}{\log\|x^{(r)}-x^*\|}$. This definition is equivalent to (4.70).

[20]The discussion more accurately concerns the norm of the error $\|e^{(r)}\| = \|x^{(r)} - x^*\|$ rather than vector $x^{(r)}$. Since, however, $\|x^{(r)} - x^*\| \geq |x_i^{(r)} - x_i^*|$ for all i, then the derivations pertain to the error for each component of vector $x^{(r)}$.

Computational effectiveness. From the standpoint of computations not only is the order of convergence important, but so are the computer resources spent on computing vector $x^{(r)}$ by an iterative or recurrent formula. These expenditures are made up of the number of operations required for computing vector function φ_f. In most methods function φ_f is constructed on the basis of computations of functions f(x) of the original equation system and of its derivatives. In the case of complex f(x) by far the greatest amount of computer time is spent on its computation (particularly for solving boundary problems, when f(x) is found by numerical integration). If θ is the number of computations of vector function f(x) and of its derivatives, required for computing the iterative functional φ_f, then the computer effectiveness $E(\varphi_j, f)$ of a method of order $v(v > 1)$ is given by the expression

$$E_v(\varphi_j, f) = v^{1/\theta}. \tag{4.73}$$

The computational effectiveness of two numerical methods of order v_1 and $v_2(v_1 > 1, v_2 > 1)$, using iterative functions ϕ_1 and ϕ_2, can be compared on the basis of (4.73). Let both methods begin calculation from the same initial approximation $x^{(0)}$ and conclude the computations after reaching the same number of correct decimal points in the solution. Then the ratio of the number I_1 required by the first method to achieve the prescribed accuracy to the required number I_2 of iteration of the second method is approximately

$$\frac{I_1}{I_2} \approx \frac{\log v_2}{\log v_1}, \tag{4.74}$$

and the ratio of summary expenses $\theta_{\Sigma 1}$ and $\theta_{\Sigma 2}$ of the two methods is approximately

$$\frac{\theta_{\Sigma_1}}{\theta_{\Sigma_2}} \approx \frac{\theta_1 \log v_2}{\theta_2 \log v_1}, \tag{4.75}$$

where $\theta_i (i = 1, 2)$ are the costs of computer operation (for computing f(x) and its derivatives) per iteration for the first and second numerical methods, respectively. /308

Range of convergence. In the general case the convergence of iterative sequence $\{x^{(r)}\}$ to solution x* of system f(x) = 0 depends both on the choice of initial approximation $x^{(0)}$ (or $x^{(0)}, x^{(1)}, \ldots, x^{(p)}$ in the case of recurrent sequence), and on the choice of method, i.e., of function φ_f. For each solution x* of system (f(x) = 0 the set of all elements $x^{(0)}$ for which sequence $\{x^{(r)}\}$, constructed with the aid of function φ_f, converges on x*, is defined as the range of convergence of point x* for method φ_f. The solution

x^* is called the point of gravitation of method φ_f if some vicinity of point x^* is entirely included in the range of convergence of this point for method φ_f. The solution x^* is called the point of repulsion of method φ_f if some vicinity of it contains not a single point of the range of convergence (with the exception of point x^* itself). In applied problems with complex computable functions $f(x)$, in cases when an initial approximation, sufficiently close to x^* is unknown, it is advisable to choose a method with a sufficiently large range of convergence (even at the cost of slow convergence).

Stages of solution. The general strategy for finding the solutions of the system

$$f(x) = 0$$

can be outlined as the following steps.

1. Determine the range of permissible $x^*: D \subset X$.

2. In range D, on the basis of all available information (the physical meaning of the problem, rough simplified solutions, solutions of analogous or similar problems, etc.), select the initial approximation $x^{(0)}$.

3. Choose the numerical method which, in terms of a priori theoretical or experimental information for the given problem, has a large range of convergence. With such methods it is advisable to make certain that systematic improvement of successive approximations is possible (for example, by evaluating the behavior of the norm of the vector of discrepancies $\|f(x^{(r)})\|$ from iteration to iteration).

4. In a sufficient proximity of the solution (for small norms of the vector of discrepancies), convert to a method with a higher order of convergence.

5. After finding one of the solutions with satisfactory accuracy, change the initial approximation and repeat the entire solution process. As the result of the new attempt either old solution x^* or different solution x^{**} will be obtained. Unfortunately, methods for finding all solutions of the system are not well enough developed and the method of changing the initial approximation is about the only practical recommendation (which, of course, does not completely guarantee success in the case of an ambiguous solution). /309

4.2.2. Simple Iteration Method

Purpose. The method is designed for solving nonlinear algebraic and transcendental equation systems $f(x) = 0$, which can be reduced to the form $x = \varphi(x)$. This is usually possible in the case when the right side $f(x)$ of the equation system is given analytically and makes it possible to separate one of the variables from each equation and transpose it to the left side.

If such separation of variables is not possible special methods, some of which are presented below [see equation (4.78)], are used. This method differs from simple computation and, if it converges, the rate of convergence is not less than linear.

Initial data. Let be given system n of nonlinear equations with n unknowns:

- in scalar form

$$f_i(x_1, x_2, x_3, \ldots, x_n)=0 \qquad (i=1, 2, 3, \ldots, n); \tag{4.76}$$

- in vectorial form

$$\mathbf{f}(\mathbf{x})=0, \tag{4.76'}$$

where

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}; \quad \mathbf{f}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\mathbf{x}) \end{bmatrix}$$

Let this system be reduced to the equivalent (see 4.2.1) system:

- in scalar form

$$x_i=\varphi_i(x_1, x_2, x_3, \ldots, x_n) \qquad (i=1, 2, \ldots, n); \tag{4.77}$$

- in vectorial form

$$\mathbf{x}=\varphi(\mathbf{x}). \tag{4.77'}$$

Required: to find real vector x*, which is one of the solutions of equation system (4.76).

The idea of the method is to determine somehow the vector of initial approximation $\mathbf{x}^{(0)}$, "sufficiently" close to the desired solution. Then, using equation (4.77'), find successive values of $\mathbf{x}^{(r)}$, by substituting into the right side of equation (4.77') the values of $\mathbf{x}^{(r-1)}$ found previously. When certain conditions imposed on the properties of the function $\varphi(\mathbf{x})$ are satisfied, this process will converge and lead to the solution x* of the initial system (4.76). /310

Computing method. The simple iteration process should be done as follows.

1. Select some initial approximation $x^{(0)}$.

2. Substitute the vector $x^{(0)}$ in the right side of equation (4.77'). The result is a new approximation, i.e., vector $x^{(1)}$:

$$x^{(1)}=\varphi(x^{(0)}).$$

3. Substitute vector $x^{(1)}$ in the right side of equation (4.77'). The result is the next (second) approximation, i.e., vector $x^{(2)}$:

$$x^{(2)}=\varphi(x^{(1)}).$$

4. Repeat this cyclic process of computing approximations

$$x^{(r+1)}=\varphi(x^{(r)}) \qquad (r=0, 1, 2, 3, \ldots)$$

until the successive approximations come together. The criterion of convergence is validity of the condition

$$\|x^{(r+1)}-x^{(r)}\|_p \leqslant \varepsilon,$$

where ε is the prescribed solution accuracy and $||x^{(r+1)} - x^{(r)}||_p$ is any p-norm of vector $x^{(r+1)} - x^{(r)}$.

The above successive approximations are formed by means of "iterations in terms of the coordinate set". If components ϕ_i of vector φ can be computed independently of each other the simple iterations method permits iterations for individual coordinates (Sidel's method). In this case:

1. Select some initial approximation $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$.

2. Compute the first approximation $x_1^{(1)}$ for the first coordinate x_1:

$$x_1^{(1)}=\varphi_1(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}),$$

and then compute the first approximation for the second coordinate x_2:

$$x_2^{(1)}=\varphi_2(x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)}),$$

for the third x_3:

$$x_3^{(1)}=\varphi_3(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, x_4^{(0)}, \ldots, x_n^{(0)})$$

and so on all the way to x_n:

$$x_n^{(1)}=\varphi_n(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_{n-1}^{(1)}, x_n^{(0)}).$$

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3. Repeat the cyclic process, computing $x^{(r)}$ by the equation

$$x_i^{(r+1)} = \varphi_i\left(x_1^{(r+1)}, x_2^{(r+1)}, \ldots, x_{i-1}^{(r+1)}, x_i^{(r)}, x_{i+1}^{(r)}, \ldots, x_n^{(r)}\right)$$
$$(i = 1, 2, 3, \ldots, n; \quad r = 0, 1, 2, \ldots)$$

until the conditions of convergence are satisfied.

Comments. 1. The convergence and rate of convergence of the simple iteration method in terms of the coordinate set depend on the method of convergence of initial system $f(x) = 0$ to equivalent system $x = \varphi(x)$. In particular, both the method of separation of coordinate x_j from the equation $x_i(\underline{x}) = 0$ and the fact that a component (index j) is separated from the equation $f_i(x) = 0$ have a considerable effect. In this regard there are no general recommendations, and in practice such conversion is done by the trial and error method. In a case of iteration in terms of individual coordinates, not only the above mentioned factors, but also the order in which the individual components of vector x are computed (i.e., enumeration of variables x_i) has an influence on computational effectiveness.

2. The simple iteration method can be generalized through more complex formulation of the equivalent system (i.e., function $\varphi_f(x)$. This sometimes makes it possible to achieve a rate of convergence higher than linear. Some formulas for the construction of functions $\varphi_f(x)$, corresponding to various generalizations of the simple iteration method, are given in Table 4.13. For more detailed description of methods based on the use of $\varphi_f(x)$ from Table 4.13, see sections 4.2.3 and 4.2.4. It should be pointed out that most iteration methods can be described through the function

$$\varphi_f(x) = x + \alpha S(x) f(x) \equiv x + \alpha p(x), \qquad (4.78)$$

where S(x) is a nondegenerate matrix; p(x) is a vector.

Definition of matrix S(x) or vector p(x) is given in Table 4.13.

The following symbols are used in the table: $x^{(0)}$ is the initial approximation, r is the number of iteration,

$$f'(x) = \frac{\partial f}{\partial x} = \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} =$$
$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots, & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, & \ldots, & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is Jacobi's matrix of vector-function f(x) of vector argument x; α, μ are real numbers. /312

TABLE 4.13. SOME FORMULAS OF GENERAL ITERATION METHODS $x^{(r+1)} = \phi_f(x^{(r)})$ FOR SOLUTION OF NONLINEAR EQUATION SYSTEMS OF THE FORM $f(x) = 0$.

No.	Iteration Equation	Method	Comment
1	$\varphi_f = x - [f'(x)]^{-1} f(x)$	Newton's standard method.	If the method converges on a simple solution, the rate of convergence is less than quadratic
2	$\varphi_f = x - [f'(z_r)]^{-1} f(x)$, where z_r is a known sequence of points	Newton's simplified method (when $z_r \equiv x^{(0)}$ it corresponds to Newton's modified method for all r).	If the method converges the rate of convergence is not less than linear.
3	$\varphi_f = x - \alpha_r [f'(x)]^{-1} f(x)$ $\alpha_r = \arg\min_{\alpha>0} \| f(x^{(r)} - \alpha \times$ $\times [f'(x^{(r)})]^{-1} f(x^{(r)}) \|^2$	Newton's expanded standard method. ($\alpha_r \to 1$ for $r \to \infty$)	Has expanded (compared with Newton's standard method) range of convergence. The rate of convergence is not less than linear
4	$\varphi_f = x - [f'(x)]^{+} f(x)$, for $[A]^{+}$ - generalized inverse matrix for initial matrix A.	Newton's generalized method (may be used in cases when the number of equations m is not equal to the number of unknowns n)	Method converges in general case on solution of system $[f'(x)]^T f(x) = 0$. If $m = n$ and $f'(x^*)$ is nondegenerate, then the method coincides with Newton's standard method.
5	$\varphi_f = x - \mu_r [f'(x)]^T f(x)$ $\mu_r = \arg\min_{\mu>0} \| f(x^{(r)} - \mu \times$ $\times [f'(x)]^T f(x) \|^2$	Method of steepest descent.	Rate of convergence linear.

TABLE 4.13 (CONTINUED)

No.	Iteration Equation	Method	Comment
6	$\varphi_{fi} = x_i - f_i(x) \dfrac{1}{\dfrac{\partial f_i(x)}{\partial x_i}}$ or $\varphi_{fi} = x_i - f_i(x) \dfrac{\alpha_r}{\dfrac{\partial f_i(x)}{\partial x_i}}$ $\alpha_r = \arg\min_{\alpha>0} \left\| \mathbf{f}\left(x_1 - \dfrac{\alpha}{f'_{11}} f_1, \ldots, x_n - \dfrac{\alpha}{f'_{nn}} f_n\right)\right\|^2$ $(\alpha = 0)$		Rate of convergence not less than linear. The advantage of the method is that it is not necessary to solve linear equation system. /313

Convergence and accuracy of method. The general theorems of convergence of iterative processes give sufficient conditions under which solution x* of system (4.76) exists, and also under conditions which convergence of iterations is guaranteed. These theorems establish that the rate of convergence of the simple iterations method is not less than linear. Estimation of the rate of convergence of the simple iteration method is of the form

$$\|\mathbf{x}^{(r+1)} - \mathbf{x}^*\| \leqslant q \|\mathbf{x}^{(r)} - \mathbf{x}^*\|, \qquad (4.79)$$

where $0 < q < 1$.

Theorems of convergence state that if functions $\phi(x)$ are differentiable in some convex vicinity G of solution x* and their first derivatives are continuous at point x*, then the sufficient condition of convergence of the iteration method, starting from the immediate vicinity of the solution, consists in the validity of the inequality

$$\lambda(J) < 1, \qquad (4.80)$$

where $\lambda(J)$ is the maximum eigen value of the Jacobi matrix J of the vector function ϕ at point x* (i.e., $\lambda(J) = \max_{\nu} |\lambda_\nu|$ $(\nu = 1, 2, \ldots, n)$.

Thus the point x* is the point of gravitation, i.e., there is a vicinity of point x* such that if the initial approximation is located in this vicinity the iteration method converges on x*. It is often possible to use instead of /314
$\lambda(J) < 1$ one of the weaker sufficient conditions of convergence:

1) $\sum_{j=1}^{n} \left|\frac{\partial \varphi_i(x)}{\partial x_j}\right| < 1 \ (i = 1, 2, \ldots, n)$ for all x of some vicinity of point x*,

or

2) $\sum_{i=1}^{n} \left|\frac{\partial \varphi_i(x)}{\partial x_j}\right| < 1 \ (j = 1, 2, \ldots, n)$ for all x of some vicinity of point x*.

However, for each specific form of equivalent system, estimates of the rate of convergence may be better. Thus, the rate of convergence according to Newton's method will not be less than quadratic.

Example 7. Calculation of required thrust P and angle of attack α by simple iteration method for equilibrium balance flight mode. Such flight, in the plane of a great circle (see example 8, Section 4.2.3) is described by the nonlinear equation system

$$f_1(P, \alpha) \triangleq P\cos(\alpha + \varphi_p) - \overline{B}_\alpha q_0 S\alpha^2 - c_{x0} q_0 S - m_0 g(h_0)\sin\theta_0 = 0,$$

$$f_2(P, \alpha) \triangleq P\sin(\alpha + \varphi_p) + c_y^\alpha q_0 S\alpha - m_0\left[g(h_0) - \frac{v_0^2}{R_3 + h_0}\right]\cos\theta_0 = 0,$$

where φ_p, $\overline{B}_\alpha$, q_0, c_{x0}, m_0, h_0, θ_0, c_y^α, S, R_E are given; P, α are unknowns to be determined; $q_0 = \frac{1}{2}\rho(h_0)\, v_0^2$; $\rho(h_0)$ corresponds to the standard atmosphere SA-64; $g(h_0) = g_0 (R_E/R_E + h_0)^2$. We write this system in a form convenient for application of the simple iteration method. For this purpose we separate variable P from the first equation and variable α from the second:

$$P = [c_{x0} q_0 S + m_0 g(h_0)\sin\theta_0 + \overline{B}_\alpha q_0 S\alpha^2]/\cos(\alpha + \varphi_p) \triangleq \varphi_1(\alpha);$$

$$\alpha = \left[m_0\left(g(h_0) - \frac{v_0^2}{R_3 + h_0}\right)\cos\theta_0 - P\sin(\alpha + \varphi_p)\right] \Big/ c_y^\alpha q_0 S \triangleq \varphi_2(P, \alpha)$$

Let the equilibrium flight mode be characterized by the parameters h_0 = 12 km; v_0 = 500 m/sec; θ_0 = 12°; g_0 = 9.80665 m/sec^2; R_E = 6,371,210 m and let the following aircraft characteristics be used:

$$\varphi_p = 3°;\ G_0 = m_0 g_0 = 15\,500 \text{ kg};\ S = 50 \text{ m}^2,\ c_{x0} = 0.01038;\ c_y^\alpha = 2.151 \text{ rad}^{-1};$$
$$\overline{B}_\alpha = 2.044 \text{ rad}^{-2}.$$

For the above flight conditions the angles of attack α in the equilibrium mode should be small.

Therefore $\alpha^{(0)} = 0$ can be used as the initial approximation for the angle of attack. In view of the fact that variable P does not enter in function ϕ_1

it is convenient to use the method of simple iteration in terms of individual coordinates:

$$\begin{cases} P^{(1)} = \varphi_1(\alpha^{(0)}), \\ \alpha^{(1)} = \varphi_2(P^{(1)}, \alpha^{(0)}), \end{cases} \qquad \begin{cases} P^{(2)} = \varphi_1(\alpha^{(1)}), \\ \alpha^{(2)} = \varphi_2(P^{(2)}, \alpha^{(1)}) \end{cases}$$

For the assumed numerical parameters the functions $\phi_1(\alpha)$ and $\phi_2(P, \alpha)$ become /315

$$\varphi_1(\alpha) = \frac{5273.9 + 123.7\alpha^2}{\cos(\alpha + 3^\circ)};$$

$$\varphi_2(P, \alpha) = \frac{15043.8 - P\sin(\alpha + 3^\circ)}{7460.4}$$

Calculation with a desk calculator gives the following results, with an accuracy up to four significant figures):

$\alpha^{(0)} = 0^\circ$; $P^{(1)} = 5287$ kg; $P^{(2)} = 5774$ kg;
$\alpha^{(1)} = 1.967^\circ$; $\alpha^{(2)} = 1.949^\circ$.
$P^{(3)} = 5766$ kg; $P^{(4)} = 5766$ kg;
$\alpha^{(3)} = 1.949^\circ$; $\alpha^{(4)} = 1.950^\circ$;

4.2.3. Newton's Method[21]

Purpose. The method is designed for solving nonlinear algebraic and transcendental equations systems $\mathbf{f}(\mathbf{x}) = 0$ with differentiable functions $\mathbf{f}(\mathbf{x})$. The method is used, in particular, for solving nonlinear equation systems encountered in the solution of boundary problems. If the iteration process converges then its rate of convergence in the vicinity of the simple (non-multiple) solution is not less than quadratic.

Initial data. Let be given a system of n equations with n unknowns:

- in scalar form

$$f_i(x_1, x_2, x_3, \ldots, x_n) = 0 \qquad (i = 1, 2, 3, \ldots, n); \tag{4.81}$$

- in vectorial form

$$\mathbf{f}(\mathbf{x}) = 0, \tag{4.81'}$$

[21] Other names are the Newton-Raphson method and linearization method.

where

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{f}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ f_n(\mathbf{x}) \end{bmatrix}$$

Required: to find vector x* which is the solution of system (4.81).

The idea of the method is to determine somehow the approximation $\mathbf{x}^{(r)}$. Then the next (r + 1) approximation is found by solving a system of linear equations. These equations are found by Taylor linerization in the vicinity of the desired solution of the left sides of equation (4.81). /317

Computing method. Equations by which the approximations of Newton's method are constructed in different varieties are presented in Table 4.14.

TABLE 4.14. ITERATION EQUATIONS OF DIFFERENT VERSIONS OF NEWTON'S METHOD

No.	Method	Iteration equation for (r + 1)-th approximations	Comments
1	Newton's standard method.	$\mathbf{x}^{(r+1)}=\mathbf{x}^{(r)}-$ $-[\mathbf{f}'(\mathbf{x}^{(r)})]^{-1}\mathbf{f}(\mathbf{x}^{(r)})$	The computations are quite convenient but require with each approximation some computation of the inverse matrix $[\mathbf{f}'(\mathbf{x}^{(r)})]^{-1}$ and multiplication of it by the vector $\mathbf{f}(\mathbf{x}^{(r)})$.
2	Newton's expanded standard method.	$\mathbf{x}^{(r+1)}=\mathbf{x}^{(r)}-$ $-\alpha_r[\mathbf{f}'(\mathbf{x}^{(r)})]^{-1}\mathbf{f}(\mathbf{x}^{(r)})$ $\alpha_r=\arg\min\limits_{1\geq\alpha\geq 0}\|\mathbf{f}(\mathbf{x}^{(r)}-\alpha\times$ $\times[\mathbf{f}'(\mathbf{x}^{(r)})]^{-1}\mathbf{f}(\mathbf{x}^{(r)}))\|^2$	
3	Newton's simplified (modified) method.	$\mathbf{x}^{(r+1)}=\mathbf{x}^{(r)}-$ $-[\mathbf{f}'(\mathbf{x}^{(0)})]^{-1}\mathbf{f}(\mathbf{x}^{(r)})$	1. Matrix $\mathbf{f}'(\mathbf{x}^{(0)})$ is computed only once at the initial point. 2. The equation gives a lower rate of convergence than that of Newton's standard method. 3. The computations require considerably less computer time than Newton's standard method.

TABLE 4.14 (CONTINUED)

No.	Method	Iteration equation for (r + 1)-th approximation	Comments
4	Newton's generalized expanded method.	$x^{(r+1)}=x^{(r)}-\alpha_r [f'(x^{(r)})]^{+}\times$ $\times f(x^{(r)})$ $\alpha_r=\arg\min_{1>\alpha>0}\|f(x^{(r)}-\alpha\times$ $\times [f'(x^{(r)})]^{-1} f(x^{(r)}))\|^2$	

Comments: 1. The meaning of the symbol arg min is explained in the basic definitions.

2. Here α is a real number.

3. $[f']^{+}$ is a generalized inverse matrix for matrix f'.

In terms of the computation it is preferable to solve a linear equation system for which there are well developed methods and standard subprograms for digital computers. The equation system for the basic version of Newton's method is:

- in scalar form

$$f_l(x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)})+\sum_{j=1}^{n}\frac{\partial f_l(x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)})}{\partial x_j}(x_n^{(r+1)}-x_j^{(r)})=0$$
$$(l=1, 2, 3, \ldots, n, \quad r=0, 1, 2, 3, \ldots); \tag{4.82}$$

- in vectorial form

$$f(x^{(r)})+f'(x^{(r)})(x^{(r+1)}-x^{(r)})=0$$
$$(r=0, 1, 2, 3, \ldots), \tag{4.82'}$$

where

$$f'(x)=\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}=\begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, \ldots, & \frac{\partial f_1}{\partial x_n} \\ \ldots\ldots\ldots\ldots \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, \ldots, & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

It is convenient to solve system (4.82') relative to unknowns $(x^{(r+1)} - x^{(r)}) = \Delta x^{(r)}$. After determination of $\Delta x^{(r)}$ the new approximation is determined by the equation

$$x^{(r+1)}=x^{(r)}+\Delta x^{(r)}. \tag{4.83}$$

Determination of coefficients α_r in generalized versions of Newton's method. In the generalized versions of Newton's method the coefficients α_r are determined as the result of minimization of the function $\Phi(\alpha)$ of variable α:

$$\Phi(\alpha)=\left\|f\left(x^{(r)}-\alpha\left[f'\left(x^{(r)}\right)\right]^{-1}f\left(x^{(r)}\right)\right\|^2 \tag{4.84}$$

(where $x^{(r)}$, $f'(x^{(r)})$, $f(x^{(r)})$ are fixed), i.e.

$$\alpha_r=\arg\min_{1\geqslant\alpha\geqslant 0}\Phi(\alpha). \tag{4.85}$$

Practical computation of the minimum of function (4.85) can be done by methods based on the construction of a quadratic parabola through three points: $\Phi(0)$, $\Phi(\frac{1}{2})$, $\Phi(1)$ and subsequent determination of the point α_p of the minimum of this parabola (for $\Phi(0) - 2\Phi(\frac{1}{2}) + \Phi(1) > 0$: /318

$$\alpha_r\approx\begin{cases}1, & \text{if} \quad \alpha_p>1;\\ \alpha_p, & \text{if} \quad 0<\alpha_p<1,\end{cases} \tag{4.85'}$$

where

$$\alpha_p=\frac{3\Phi(0)-4\Phi\left(\frac{1}{2}\right)+\Phi(1)}{4\left[\Phi(0)-2\Phi\left(\frac{1}{2}\right)+\Phi(1)\right]}$$

Convergence and accuracy of method. The general theorems guarantee a quadratic rate of convergence. These theorems establish the sufficient conditions of the existence of the solution and give estimates of the error of the solution obtained. The estimate of the rate of convergence is

$$\left\|x^{(r+1)}-x^*\right\|\leqslant C\left\|x^{(r)}-x^*\right\|^2, \tag{4.86}$$

where C is a constant.

Advantages and disadvantages of the method. 1. It is a comparatively simple computing method, since there are extremely economical ways of solving linear equation systems.

2. The method has quadratic convergence in the vicinity of the solution.

3. It has several modifications.

4. All subsequent computations in the method are constructed independently of the preceding.

5. It is stable to the choice of initial approximation in some vicinity of the solution.

6. Most of the computer time is spent on computation of the left sides of the equation f(x) = 0 and their derivatives. Therefore for high-order boundary problems the method may lead to large expenditures of computer time for obtaining each new approximation.

Example 8. Determined by Newton's method the numerical parameters of the equilibrium mode of balanced flight of an aircraft. The equations of motion of the center of mass of the aircraft, executing flight without roll or skid in the plane of a great circle in the atmosphere of the spherical nonrotating Earth, are written in cartesian vertical-velocity coordinate system $Ox_V y_V z_V$ as

$$\left.\begin{aligned} m\dot{v} &= P\cos(\alpha+\varphi_p) - Q(v, h, \alpha) - mg(h)\sin\theta; \\ mv\dot{\theta} &= P\sin(\alpha+\varphi_p) + Y(v, h, \alpha) - m\left[g(h) - \frac{v^2}{R_3+h}\right]\cos\theta; \\ \dot{h} &= v\sin\theta; \quad \dot{L} = \frac{v\cos\theta}{R_3+h}R_3 \end{aligned}\right\} \quad \text{(I)} \qquad \text{(I)}$$

(for the symbols see example 3 in Section 4.1.1). Since we are examining equilibrium flights, the equation of motion relative to the center of mass need not be considered. The conditions of equilibrium of the flight mode mean that for each fixed altitude $h = h_0$ the identities $\dot{v} \equiv 0$; $\dot{\theta} \equiv 0$ are satisfied. In this case the mass of the aircraft is considered constant.

Since altitude h for equilibrium modes is considered fixed and the right sides of equations (I) do not depend on range L, the equations for $\dot{h}$ and $\dot{L}$ do not take part in the determination of the parameters of equilibrium modes.

Thus the parameters of the equilibrium flight mode are determined by the following system of two nonlinear transcendental equations, relating six independent variables: P, α, v, h, θ, m:

$$\left.\begin{aligned} f_1(P, \alpha, v, \theta, h, m) &= P\cos(\alpha+\varphi_p) - \\ &- Q(v, h, \alpha) - mg(h)\sin\theta = 0; \\ f_2(P, \alpha, v, \theta, h, m) &= P\sin(\alpha+\varphi_p) + \\ &+ Y(v, h, \alpha) - m\left[g(h) - \frac{v^2}{R_3+h}\right]\cos\theta = 0 \end{aligned}\right\} \quad \text{(II)}$$

(ϕ_p and R_E are assumed to be known).

If any four of the six independent variables are also known, then the other two can be found by means of numerical solution of this equation system. Let, for example, be given v_0, θ_0, h_0, m_0. We are required to determine the required thrust P and angle of attack α of the equilibrium flight mode. Then equation system (II) with two unknowns P and α is written as:

$$\left.\begin{aligned} f_1(P, \alpha) &\triangleq P\cos(\alpha+\varphi_P) - \bar{B}_\alpha(M)\,Sq_0\alpha^2 - \\ &- c_{x0}q_0S - m_0g(h_0)\sin\theta_0 = 0; \\ f_2(P, \alpha) &\triangleq P\sin(\alpha+\varphi_P) + c_y^\alpha q_0S\alpha - \\ &- m_0\left[g(h_0) - \frac{v_0^2}{R_3+h}\right]\cos\theta_0 = 0 \end{aligned}\right\} \qquad \text{(III)}$$

(in this system it is assumed that

$$Q = c_{x0}q_0S + \bar{B}_\alpha q_0S\alpha^2; \quad Y = c_y^\alpha q_0S\alpha).$$

When solving system (III) by means of the standard or simplified Newton methods it is necessary to assign the initial approximations $P^{(0)}$, $\alpha^{(0)}$ after which the first approximations $P^{(1)}$, $\alpha^{(1)}$ are found from the solution of the linearized system (4.82):

$$\frac{\partial f_1(P^{(0)}, \alpha^{(0)})}{\partial P}(P^{(1)}-P^{(0)}) + \frac{\partial f_1(P^{(0)}, \alpha^{(0)})}{\partial \alpha}(\alpha^{(1)}-\alpha^{(0)}) = -f_1(P^{(0)}, \alpha^{(0)});$$
$$\frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial P}(P^{(1)}-P^{(0)}) + \frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial \alpha}(\alpha^{(1)}-\alpha^{(0)}) = -f_2(P^{(0)}, \alpha^{(0)}),$$

where

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$$\frac{\partial f_1(P^{(0)}, \alpha^{(0)})}{\partial P} = \cos(\alpha+\varphi_P); \quad \frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial P} = \sin(\alpha^{(0)}+\varphi_P);$$
$$\frac{\partial f_1(P^{(0)}, \alpha^{(0)})}{\partial \alpha} = -P^{(0)}\sin(\alpha^{(0)}+\varphi_P) - 2\bar{B}_\alpha q_0S\alpha^{(0)};$$
$$\frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial \alpha} = P^{(0)}\cos(\alpha^{(0)}+\varphi_P) + c_y^\alpha q_0S;$$
$$f_1(P^{(0)}, \alpha^{(0)}) = P^{(0)}\cos(\alpha^{(0)}+\varphi_P) - \bar{B}_\alpha q_0S(\alpha^{(0)})^2 - c_{x0}q_0S - m_0g(h_0)\sin\theta_0 \neq 0;$$
$$f_2(P^{(0)}, \alpha^{(0)}) = P^{(0)}\sin(\alpha^{(0)}+\varphi_P) + c_y^\alpha q_0S\alpha^{(0)} - m_0\left[g(h_0) - \frac{v^2_0}{R_3+h_0}\right]\cos\theta_0 \neq 0$$

(in the general case these values are not equal to zero in view of arbitrary initial approximations $P^{(0)}$, $\alpha^{(0)}$).

The subsequent iterations (r = 2, 3, ...) are constructed by means of the analogous system as follows:

- for Newton's standard method

$$\left.\begin{aligned} &\frac{\partial f_1(P^{(r)}, \alpha^{(r)})}{\partial P}(P^{(r+1)}-P^{(r)}) + \frac{\partial f_1(P^{(r)}, \alpha^{(r)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)}) = \\ &\qquad = -f_1(P^{(r)}, \alpha^{(r)}); \\ &\frac{\partial f_2(P^{(r)}, \alpha^{(r)})}{\partial P}(P^{(r+1)}-P^{(r)}) + \frac{\partial f_2(P^{(r)}, \alpha^{(r)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)}) = \\ &\qquad = -f_2(P^{(r)}, \alpha^{(r)}); \end{aligned}\right\} \qquad \text{(IV)}$$

- Newton's simplified method

$$\left.\begin{aligned}
&\frac{\partial f_1(P^{(0)}, \alpha^{(0)})}{\partial P}(P^{(r+1)}-P^{(r)})+\frac{\partial f_1(P^{(r)}, \alpha^{(0)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)})= \\
&\qquad =-f_1(P^{(r)}, \alpha^{(r)}); \\
&\frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial P}(P^{(r+1)}-P^{(r)})+\frac{\partial f_2(P^{(0)}, \alpha^{(0)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)})= \\
&\qquad =-f_2(P^{(r)}, \alpha^{(r)});
\end{aligned}\right\} \qquad (V)$$

- Newton's expanded method (Newton's steepest descent method)

$$\left.\begin{aligned}
&\frac{\partial f_1(P^{(r)}, \alpha^{(r)})}{\partial P}(P^{(r+1)}-P^{(r)})+\frac{\partial f_1(P^{(r)}, \alpha^{(r)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)})= \\
&\qquad =-\lambda_r f_1(P^{(r)}, \alpha^{(r)}); \\
&\frac{\partial f_2(P^{(r)}, \alpha^{(r)})}{\partial P}(P^{(r+1)}-P^{(r)})+\frac{\partial f_2(P^{(r)}, \alpha^{(r)})}{\partial \alpha}(\alpha^{(r+1)}-\alpha^{(r)})= \\
&\qquad =-\lambda_r f_2(P^{(r)}, \alpha^{(r)}),
\end{aligned}\right\} \qquad (VI)$$

where λ_r is either a known number ($\lambda_r < 1$), or is found from the condition /321

$$\left.\begin{aligned}
\lambda_r = \arg \min_{\lambda>0}\{&f_1^2[P^{(r)}+\lambda(P^{(r+1)}-P^{(r)}), \alpha^{(r)}+ \\
&+\lambda(\alpha^{(r+1)}-\alpha^{(r)})]+f_2^2[P^{(r)}+\lambda(P^{(r+1)}-P^{(r)}), \\
&\alpha^{(r)}+\lambda(\alpha^{(r+1)}-\alpha^{(r)})]\} = \arg\min_{\lambda>0}[f_1^2(\lambda)+f_2^2(\lambda)].
\end{aligned}\right\} \qquad (VII)$$

The minimum of the function $[f_1^2(\lambda) + f_2^2(\lambda)]$ of descent parameter λ is found with the aid of system (4.85'). Assuming the differences $P^{(r+1)} - P^{(r)}$ and $\alpha^{(r+1)} - \alpha^{(r)}$ to be unknown increments of $\Delta P^{(r)}$ and $\Delta\alpha^{(r)}$, we solve systems (IV)-(VI) relative to them. After determination of the increments $\Delta P^{(r)}$ and $\Delta\alpha^{(r)}$ the new approximation $P^{(r+1)}$, $\alpha^{(r+1)}$ is found with the aid of equations (4.83):

$$P^{(r+1)} = P^{(r)} + \Delta P^{(r)};$$
$$\alpha^{(r+1)} = \alpha^{(r)} + \Delta\alpha^{(r)}.$$

The results of calculation of parameters P, α of the equilibrium flight mode by Newton's standard and simplified methods are presented in Tables 4.15 a and 4.15 b. The following numerical parameters were used in the calculations:

$$G_0 = m_0 g_0 = 15\,500 \text{ kg}; \quad S = 50 \text{ M}^2; \quad \varphi_P = 3^2;$$
$$R_3 = 6\,371\,210 \text{ M}; \quad g_0 = 9.80665 \text{ m/sec}^2;$$
$$v = v_0 = 500 \text{ m/sec}, \quad \theta = \theta_0 = 12°; \quad h = h_0 = 12\,000 \text{ M};$$
$$\varrho(h_0) = 0.0318 \text{ kg} \cdot \text{sec}^2/\text{m}^4.$$

Aerodynamic forces Q and Y were computed by the equations:

$$Q = c_{x0} q_0 S + \eta c_y^{\alpha} q_0 S \alpha^2; \quad Y = c_y^{\alpha} q_0 S \alpha.$$

where the coefficients c_{x0}; $\eta c_y^{\alpha} = \overline{B}_{\alpha}$, c_y^{α} are determined by equations presented in example 3 of Section 4.1.1, and for the given flight conditions (M_0 = 1.695) are:

$$c_{x0} = 0.0104; \ \overline{B}_{\alpha} = \eta c_y^{\alpha} = 2.044 \text{ 1/rad}; \ c_y^{\alpha} = 2.151 \text{ 1/rad}.$$

The values q_0 and $g(h_0)$ are determined by the relations

$$q_0 = \frac{1}{2} \varrho(h_0) v_0^2; \ g(h_0) = g_0 \left(\frac{R_3}{R_3 + h_0} \right)^2$$

and for the given flight conditions are:

$$q_0 = 3975 \text{ kg/m}^2; \ g(h_0) = 9.77 \text{ m/sec}^2.$$

For the parameters of motion and aircraft characteristics presented herein equations (III) acquire the form

$$\left.\begin{aligned} f_1(P, \alpha) &= P \cos(\alpha + 3^\circ) - 123.5\alpha^2 - 5274 = 0; \\ f_2(P, \alpha) &= P \sin(\alpha + 3^\circ) + 7450\alpha - 15\,089 = 0. \end{aligned}\right\} \qquad \text{(VIII)}$$

Two calculations were done for two different initial approximations for P and α. The first variation $(P^{(0)}, \alpha^{(0)}) = (0, 0)$ was quite arbitrary, but the second $((P^{(0)}, \alpha^{(0)} = (5799.1 \text{ kg}, 2.0165^\circ))$ was found by means of approximate methods of flight dynamics. It is known that for the given flight conditions of the examined type of aircraft the angles of attack α are quite small and it may be assumed without too much error that $\sin(\alpha + \phi_p) \approx \alpha + \phi_p$, and that thrust P and drag Q are magnitudes of the same order of smallness. Ignoring the product P sin $(\alpha + \phi_p)$, which is approximately one order of magnitude smaller than the other terms, we find from the second equation (III) the approximate angle of attack α_{equi} in the equilibrium flight mode: /322

$$\alpha^{(0)} \approx \alpha_{equi} \approx \frac{m_0}{c_y^{\alpha} q_0 S} \left[g(h_0) - \frac{v_0^2}{R_3 + h_0} \right] = 2.01648373^\circ,$$

and from the first equation we find the approximate thrust:

$$P^{(0)} \approx P_{equi} = \frac{\left[c_{x0} + \eta c_y^{\alpha} (\alpha^{(0)})^2 \right] q_0 S + m_0 g(h_0) \sin \theta_0}{\cos(\alpha^{(0)} + \varphi_p)} = 5799.12677 \text{ kg}.$$

TABLE 4.15 a. CALCULATION OF PARAMETERS OF EQUILIBRIUM FLIGHT MODE OF AIRCRAFT BY MEANS OF NEWTON'S STANDARD METHOD USING HIGH SPEED DIGITAL COMPUTERS

Step h_α of numerical differentiation with respect to α of function $f_i\ (P,\alpha)$, $i = 1, 2$, is $h_\alpha = 0.1°$,
Step h_P of numerical integration with respect to P is $h_P = 100$ kg.
(The printout results of the computer are presented in this table without processing)

Variation of Initial Conditions	No. of Iteration r	$P^{(r)}$ kg	$\alpha^{(r)}$ deg	$f1(P^{(r)}, \alpha^{(r)})$ kg	$f2(P^{(r)}, \alpha^{(r)})$ kg	$\|\|f(P^{(r)}, \alpha^{(r)})\|\|$ kg	Total No. of Computations of vector f for r iterations
1	0	0	0	—5273,93121	—15043,7867.	15941;4513	1
	1	5305,68561	1,97926349	—472,854079	182,829877	506,969175	4
	2	5765,23037	1,94977501	—0,454937339	—0,235895633	0,51245949	7
	3	5765,70007	1,94980085	$0{,}321805477 \cdot 10^{-3}$	$0{,}476837158 \cdot 10^{-6}$	$0{,}321805830 \cdot 10^{-3}$	10
	4	5765,69975	1,94980086	$0{,}119209289 \cdot 10^{-6}$	$0{,}238418579 \cdot 10^{-6}$	$0{,}266560075 \cdot 10^{-6}$	13
	5	5765,69975	1,94980086	0	0	0	16
2	0	5799,12677	2,01648373	0	507,089209	507,089209	1
	1	5764,31732	1,94982186	—1,38757956	$0{,}394866466 \cdot 10^{-1}$	1,38814128	4
	2	5765,69949	1,94980086	$-0{,}261545181 \cdot 10^{-3}$	$-0{,}476837158 \cdot 10^{-6}$	$0{,}261545616 \cdot 10^{-3}$	7
	3	5765,69975	1,94980086	0	$0{,}238418579 \cdot 10^{-6}$	$0{,}238418579 \cdot 10^{-6}$	10
	4	5765,69975	1,94980086	0	0	0	13

Note: Commas indicate decimal points.

TABLE 4.15 b. CALCULATION OF PARAMETERS OF EQUILIBRIUM FLIGHT MODE OF AIRCRAFT BY NEWTON'S SIMPLIFIED METHOD USING HIGH SPEED DIGITAL COMPUTER

Step h_α of numerical differentiation with respect to α functions f_i (P, α), $(i = 1, 2$, is $h_\alpha = 0.1°$; the analogous step h_p of differentiation with respect to P is $h_p = 100$ kg. (The printout results of the computer are presented in the table without processing.)

Variation of Initial Conditions	No. of Iteration r	$P^{(r)}$ kg	$\alpha^{(r)}$ deg	$f_1(P^{(r)}, \alpha^{(r)})$ kg	$f_2(P^{(r)}, \alpha^{(r)})$ kg	$\|\|f(P^{(r)}, \alpha^{(r)}\|\|$ kg	Total No. of Computations of vector f = $= (f_1, f_2)$ for r iterations.
1	0	0	0	—5273,93121	—15043,7867	15941,4513	1
	1	5305,68561	1,97926349	—472,854079	182,829877	506,969175	4
	2	5778,84394	1,95143751	12,2911077	13,5086834	18,2635116	5
	3	5766,51460	1,94971329	0,854811848	—0,591781140	1,03966735	6
	4	5765,65967	1,9497986l	—0,388243198·10—1	—0,204672813·10—1	0,438889213·10—1	7
	5	5765,69857	1,94980108	—0,127428770·10—2	0,156855583·10—2	0,202093457·10—2	8
	6	5765,69985	1,94980086	0,101864338·10—3	0,216960907·10—4	0,104149237·10—3	9
	7	5765,69975	1,94980086	0,119209289·10—5	—0,405311584·10—5	0,4	10
	8	5765,69975	1,94980086	—0,238418579·10—6	0,238418579·10—6	0,3	11
	9	5765,69975	1,94980086	0	0		12
2	0	5799,12677	2,01648373	0	507,089209	507,089209	1
	1	5764,31732	1,94982186	—1,38757956	0,394866466·10—1	1,38814128	4
	2	5765,69917	1,94980065	—0,477671623·10—3	—0,158977508·10—2	0,165998644·10—2	5
	3	5765,69976	1,94980086	0,602006913·10—5	—0,238418579·10—6	0,602478844·10—5	6
	4	5765,69965	1,94980086	0	—0,476837158·10—6	0,476837158·10—6	7
	5	5765,69975	1,94980086	0	0	0	8

Note: Commas indicate decimal points.

Tables 4.15 a and 4.15 b demonstrate the effect of a sufficiently close initial approximation on the accuracy of the result and volume of computer operation. They show, in particular, that to find the solution with an accuracy up to four accurate decimal points for the first variation of initial approximations it is necessary in Newton's simplified method to do 7 or 8 computations of the functions $f(x)$, and in Newton's standard method, about 10. For the second variation of initial conditions, the solution is found with the prescribed accuracy by Newton's simplified method in five computations of f(x), and in seven computations by Newton's standard method. Thus, in the given example, despite the lower rate of conversions, Newton's simplified method requires fewer computations than Newton's standard method (for a result with the same order of accuracy).

We will note that in the solution of the given example, the partial derivatives $\frac{\partial f_i}{\partial P}$, $\frac{\partial f_i}{\partial \alpha}$ (i = 1, 2) will be found numerically using the difference formulas

$$\frac{\partial f_i(P, \alpha)}{\partial P} \approx \frac{f_i(P + h_P, \alpha) - f_i(P, \alpha)}{h_P};$$

$$\frac{\partial f_i(P, \alpha)}{\partial \alpha} \approx \frac{f_i(P, \alpha + h_\alpha) - f_i(P, \alpha)}{h_\alpha}.$$

where the steps of differentiation h_P and h_α were assumed to be h_P = 100 kg, h_α = 0.1°. (Experiments with other differentiation steps h_P, h_α revealed that in the given problem they do not have much effect on convergence if they are chosen within the limits

$$10 \text{ kg} \leqslant h_P \leqslant 100 \text{ kg},$$
$$0.01° \leqslant h_\alpha \leqslant 0.1°).$$

4.2.4. Method of Secants

Purpose. The method is designed for solving nonlinear algebraic and transcendental equations systems f(x) = 0 without computation of partial derivatives of the vector function f(x). The method is used, in particular, for solving equation systems encountered in the solution of boundary value problems by reducing them to the Cauchy problem. The computational structure of this method is more economical in terms of the number of computations of the function f(x) than the method of chords and Newton's method.

Initial data. Let be given a system of n equations with n unknowns:

- in scalar form

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$$f_i(x_1, x_2, \ldots, x_n) = 0 \qquad (i = 1, 2, \ldots, n), \tag{4.87}$$

- in vectorial form

$$f(x)=0, \tag{4.87'}$$

where

$$x=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \quad f(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \\ \cdots \\ f_n(x) \end{bmatrix}.$$

Required: to find vector x* which is the solution of system (4.87'). The idea of the method is, according to known (n + 1)-th initial approximations $x^{(p)} = (x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)})^T$ (p = - n, - n + 1, ..., - 1.0), to compute the left sides of (4.87) $f^{(p)} = f(x^{(p)}) = f_1(x^{(p)}, f_2(x^{(p)}), \ldots, f_n(x^{(p)}))^T$, to each value $f^{(p)}$ of which is ascribed a certain "weight" $\mu_{(p)}$, $\sum_p \mu_{(p)} = 1$[22].

The values $\mu_{(p)}$ are determined such that the center of gravity of the system of points $f^{(p)}$ (p = - n, ..., - 1.0) $\tilde{F}^n$, $f = (f_1, f_2, \ldots, f_n) \in \tilde{F}^n$, is also located at the coordinate origin of this space. Using the values found for $\mu_{(p)}$, (p = - n, ..., - 1.0), the position of the center of mass of the system of points $x^{(p)}$, (p = - n, ..., 0) is determined and is used as the new approximations.

Computing method. Let in the r-th iteration be known (n + 1) initial (reference) points (initial approximations):

$$x^{(p+r)}=(x_1^{(p+r)}, x_2^{(p+r)}, \ldots, x_n^{(p+r)}) (p=-n, -n+1, \ldots, -1, 0). \tag{4.88}$$

Then the next approximation $x^{(r+1)}$ to the solution of system (4.87') is found according to the equation for the mean weight of points $x^{(p+r)}$ with weights $\mu_{(p+r)}$:

$$x^{(r+1)}=\sum_{p=-n}^{0} \mu_{(p+r)} x^{(p+r)}. \tag{4.89}$$

Weights $\mu_{(p+r)}$ are determined from system (n + 1) of equations, linear relative to $\mu_{(p+r)}$:

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[22] The variables μ_p are sometimes called barycentric coordinates of points $f(x^{(p)}) \in \tilde{F}^n$.

$$\left.\begin{aligned}&\sum_{p=-n}^{0}\mu_{(p+r)}f_l(\mathbf{x}^{(p+r)})=0, \qquad (l=1, 2, \ldots, n;\\ &\sum_{p=-n}^{0}\mu_{(p+r)}=1,\end{aligned}\right\} \tag{4.90}$$

or in factored form: /326

$$\left.\begin{aligned}&\mu_{(-n+r)}f_1(\mathbf{x}^{(-n+r)})+\mu_{(-n+1+r)}f_1(\mathbf{x}^{(-n+1+r)})+\\ &+\ldots+\mu_{(-1+r)}f_1(\mathbf{x}^{(-1+r)})+\mu_{(r)}f_1(\mathbf{x}^{(r)})=0;\\ &\mu_{(-n+r)}f_2(\mathbf{x}^{(-n+r)})+\mu_{(-n+1+r)}f_2(\mathbf{x}^{(-n+1+r)})+\\ &+\ldots+\mu_{(-1+r)}f_2(\mathbf{x}^{(-1+r)})+\mu_{(r)}f_2(\mathbf{x}^{(r)})=0;\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\mu_{(-n+r)}f_n(\mathbf{x}^{(-n+r)})+\mu_{(-n+1+r)}f_n(\mathbf{x}^{(-n+1+r)})+\\ &+\ldots+\mu_{(-1+r)}f_n(\mathbf{x}^{(-1+r)})+\mu_{(r)}f_n(\mathbf{x}^{(r)})=0;\\ &\mu_{(-n+r)}+\mu_{(-n+1+r)}+\ldots+\mu_{(-1+r)}+\mu_{(r)}=1.\end{aligned}\right\} \tag{4.91}$$

This system is written in vectorial form as

$$F^{(r)}\mu^{(r)}=(0, 0, \ldots, 1)^{\mathrm{T}}. \tag{4.92}$$

Defined here are: $F^{(r)}$ -- matrix of system (4.90) or (4.91):

$$F^{(r)}=(\mathbf{F}_0^{(r)}, \mathbf{F}_1^{(r)}, \ldots, \mathbf{F}_n^{(r)})=$$

$$=\begin{bmatrix} f_1(\mathbf{x}^{(-n+r)}), & f_1(\mathbf{x}^{(-n+r+1)}), \ldots, & f_1(\mathbf{x}^{(r)}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(\mathbf{x}^{(-n+r)}), & f_n(\mathbf{x}^{(-n+r+1)}), \ldots, & f_n(\mathbf{x}^{(r)}) \\ 1 & 1 & 1 \end{bmatrix}; \tag{4.93}$$

$$\mathbf{F}_j^{(r)}=\begin{bmatrix} f_1(\mathbf{x}^{(-n+j+r)}) \\ f_2(\mathbf{x}^{(-n+j+r)}) \\ f_n(\mathbf{x}^{(-n+j+r)}) \\ 1 \end{bmatrix} \qquad (j=0, 1, \ldots, n; \tag{4.94}$$

$$\mu^{(r)}=(\mu_{(-n+r)}, \mu_{(-n+1+r)}, \ldots, \mu_{(-1+r)}, \mu_{(r)})^{\mathrm{T}}. \tag{4.95}$$

The solution from here is examined in vectorial form. The solution of equation (4.92) is

$$\mu^{(r)}=[F^{(r)}]^{-1}(0, 0, \ldots, 0, 1)^{\mathrm{T}}, \tag{4.96}$$

i.e., $\mu^{(r)}$ is the right hand column of inverse matrix $[F^{(r)}]^{-1}$. Using the vector $\mu^{(r)}$ we find according to equation (4.89) $\mathbf{x}^{(r+1)}$. After finding the (r + 1)-th approximation of $\mathbf{x}^{(r+1)}$, from series (n + 1) of initial points (4.88) is excluded one of points $\mathbf{x}^{(p+r)}$, $-n \leqslant p \leqslant 0$ and inserted in place of

it is the newly found point $x^{(r+1)}$. If a point with arbitrary index $(-n+m+r)$, $0 \leqslant m \leqslant n$, is excluded, the new matrix $F^{(r+1)}$ of system (4.90) or (4.91) will be

$$F^{(r+1)} = (F_0^{(r)}, F_1^{(r)}, \ldots, F_{m-1}^{(r)}, F_m^{(r+1)}, F_{m+1}^{(r)}, \ldots, F_n^{(r)}) = \qquad \text{/327}$$

$$= \begin{bmatrix} f_1(x^{(-n+r)}), \ldots, f_1(x^{(m-n-1+r)}), f_1(x^{(r+1)}), f_1(x^{(m-n+1+r)}), \ldots, f_1(x^{(r)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n(x^{(-n+r)}), \ldots, f_n(x^{(m-n-1+r)}), f_n(x^{(r+1)}), f_1(x^{(m-n+1+r)}), \ldots, f_n(x^{(r)}) \\ 1 \quad \ldots \quad 1 \quad , \quad 1 \quad , \quad 1 \quad \ldots \quad 1 \end{bmatrix}, \qquad (4.97)$$

and in place of the column

$$F_m^{(r)} = (f_1(x^{(-n+m+r)}), f_2(x^{(-n+m+r)}), \ldots, f_n(x^{(-n+m+r)}), 1)^T$$

is substituted the column

$$F_m^{(r+1)} = (f_1(x^{(r+1)}), f_2(x^{(r+1)}), \ldots, f_n(x^{(r+1)}), 1)^T.$$

Inverting matrix $F^{(r+1)}$, we find the new vector $\mu^{(r+1)} = [F^{(r+1)}]^{-1}(0, \ldots, 0, 1)^T$, and from it, using equation (4.89), the new approximation $x^{(r+2)}$, etc. The iteration process is continued until the discrepancy norm of solution (4.87') is achieved, e.g., until standard residuals $||f(x^{(r+1)})||$ becomes less than the given value $\varepsilon > 0$.

The following procedures are recommended for choosing the excluded point:

a) exclusion of points with smallest index ("most significant" point). In this case the new vector $F^{(r+1)}$ in matrix $f(x^{(r+1)})$ is always on the right and the "most significant" matrix $f(x^{(r-n+1)}$ is always on the left. The convergence of the method in this procedure may be nonmonotonic;

b) exclusion of point with maximum square of discrepancy norm. This method is often used in combination with a modification that increases the range of convergence of the method of secants.

The computational structure of the modified method of secants with exclusion of the point corresponding to the maximum square of discrepancy norm amounts to the following.

The squares of the discrepancy norm $\|f(x^{(p+r)})\|^2 = \sum_{i=1}^{n} f_i^2(x^{(p+r)})$ are computed in each iteration for all points $x^{(p+r)}$, $(r-n \leqslant p+r \leqslant r)$ and from among them is found the point $x^{(m+r)}$ at which this value is maximum:

$$\|f(x^{(m+r)})\|^2 = \max_{-n+r \leqslant p+r \leqslant r} \|f(x^{(p+r)})\|^2 = \|f_r\|^2_{\max}.$$

Then the point $x^{(l+r)}$ at which the square of the discrepancy norm is minimum is determined:

$$\|f(x^{(l+r)})\|^2 = \min_{-n+r \le p+r \le r} \|f(x^{(p+r)})\|^2 = \|f_r\|^2_{\min}.$$

To prevent divergence of the process during computation from remote initial approximations the discrepancy must be reduced gradually, and not in one step. For this purpose a system of variables $\mu_{(p+r)}$ is found, for which the "center of gravity" of a system of (p + 1) points $f^{(p+r)}$ lies not at the coordinate origin of space $\tilde{F}^n$, $f^{(p+r)} \in \tilde{F}^n$, but at some point on the ray constructed from the coordinate origin to nearest point $f^{(l+r)} = f(x^{(l+r)})$ with the value $\alpha_r^{(0)}$ of the coefficient of discrepancy reduction thus given $(0 < \alpha_r^{(0)} \leqslant 1)$, $\mu_{(p+r)}$ are found from the equation system: /328

$$\sum_{p=-n}^{0} \mu_{(p+r)} f_i(x^{(p+r)}) = (1 - \alpha_r^{(0)}) f_i(x^{(l+r)}) \qquad (i = 1, 2, \ldots, n);$$

$$\sum_{p=-n}^{0} \mu_{(p+r)} = 1. \qquad (4.98)$$

The new approximation $x^{(r+1)}$ is, as before, determined from (4.89). If $||f(x^{(r+1)})||^2 > ||f_r||^2_{\max}$, then $\alpha_r^{(1)} = \alpha_r^{(0)}/2$ is assumed and system (4.98) is solved again with the former set of reference points until the condition

$$\|f(x^{(r+1)})\|^2 < \|f_r\|^2_{\max}. \qquad (4.99)$$

is satisfied.

After condition (4.99) is satisfied $x^{(r+m)}$ and $f(x^{(r+m)})$ are replaced by new $x^{(r+1)}$ and $f(x^{(r+1)})$ and the iterations are continued until the required accuracy is achieved.

Condition (4.99) guarantees monotonicity of convergence with respect to the discrepancy. The value $\alpha_r^{(0)}$ may increase as the approximation approaches the solution: $\alpha_r^{(0)} \to 1$ for $x^{(r+1)} \to x^*$ or $||f(x^{(r+1)})|| \to 0$. When $\alpha_r = 1$ system (4.98) converts to system (4.90).

Comments. 1. Matrix $F^{(r+1)}$ differs from matrix $F^{(r)}$ by only one column. This circumstance can be used for cutting down the computer time for access

to matrix $F^{(r+1)}$. Thus, if $F^{(r+1)}$ is obtained from $F^{(r)}$ by substituting for m-th column new vector $f(x^{(r+1)})$, then the elements $[F^{(r+1)}]^{-1}_{ij}$ of inverse matrix $[F^{(r+1)}]^{-1}$ can be computed through the known elements of matrix $[F^{(r)}]^{-1}$ by the equations

$$[F^{(r+1)}]^{-1}_{ij} = [F^{(r)}]^{-1}_{ij} - [F^{(r)}]^{-1}_{mj} \frac{\Delta_i^{(r)} - \delta_{im}}{\Delta_m^{(r)}} \qquad (i, j = 1, 2, \ldots, n+1),$$

where $\Delta_i^{(r)}$, $\Delta_m^{(r)}$ are elements of column vector $\Delta^{(r)}$;

$$\Delta^{(r)} = \begin{bmatrix} \Delta_1^{(r)} \\ \Delta_2^{(r)} \\ \cdot \\ \cdot \\ \cdot \\ \Delta_{n+1}^{(r)} \end{bmatrix} = [F^{(r)}]^{-1} f(x^{(r+1)}); \quad \delta_{im} = \begin{cases} 0 & \text{for } i \neq m; \\ 1 & \text{for } i = m. \end{cases}$$

2. During the iteration process vectors $f(x^{(p+r)}$, p = - n, - n + 1, ...0, can sometimes become linearly dependent. Then the method of secants degenerates, since for such vectors the matrix $[F^{(r)}]^{-1}$ does not exist. In this case it is necessary to choose a new system of reference points $x^{(p)}$, p = - n, - n + 1, ..., 0. /329

Convergence and accuracy of method. Under rather general, but unwieldy conditions, the method of secants converges. The asymptotic rate of convergence, characterizing how many times the number of correct digits is increased in the answer after each iteration, is expressed by the equation

$$v = \lim_{r \to \infty} \frac{\log \|x^{(r+1)} - x^*\|}{\log \|x^{(r)} - x^*\|}, \tag{4.100}$$

where x* is the solution of system (4.87);
r is the number of iteration.

The value v is equal to the dominant positive root $v_0(n)$ of the characteristic equation of the method of secants

$$v^{n+1} = v^n + 1, \tag{4.101}$$

where n is the number of equations of system (4.87).

For one equation (n = 1) the asymptotic rate of convergence $v_{m.s}$ of the method of secants $v_{m.s}$ = 1.618, which is less than the asymptotic rate of

convergence of Newton's method[23] ($v_{n.m} = 2$). For arbitrary number n of equations the asymptotic rate of convergence of the method of secants is always less than the asymptotic rate of convergence of Newton's method:

$$v_{m.s}(n) < v_{n.m} = 2,$$

and when $n \to \infty$, $v_{m.s} \to 1$.

For the error of the method the following estimate is valid

$$\|x^{(r+1)} - x^*\| \leqslant C \|x^{(r)} - x^*\| \cdot \|x^{(r-n)} - x^*\|, \tag{4.102}$$

where r is the number of iteration; C is the convergence constant.

This estimate shows that the error in the (r + 1)-th iteration does not exceed a value proportional to the product of errors in the r-th and (r - n)-th iterations.

The computational effectiveness of the method is characterized not only by the rate of convergence (i.e., by how many times the number of correct digits is increased in the solution per iteration), but also by the volume of computations required for finding a certain number of correct digits. This volume in the final analysis characterizes the required computer time.

Introduced here is the effectiveness index E_m of the method, which is the number indicating how many times the error is reduced (how many times the number of correct digits is increased) in the solution $x^{(r+1)}$ as the result of one-time computation of the vector f(x) after a sufficient number of iterations: /330

$$E_M = \sqrt[\theta_m]{v_m} = v_m^{\frac{1}{\theta_m}} = \lim_{r \to \infty} \left[\frac{\log \|x^{(r+1)} - x^*\|}{\log \|x^{(r)} - x^*\|} \right]^{\frac{1}{\theta_m}}, \tag{4.103}$$

where θ_m is the number of computations of the vector f(x) per iteration;

v_m is the asymptotic rate of convergence of the method.

For the method of secants $\theta_{m.s} = 1$, and for Newton's method[24] $\theta_{n.m.} = (n + 1)$. It has been established for all n (see Table 4.16) the inequality

$$v_{m.s}(n) = E_{m.s} > E_{n.m} = 2^{1/n+1}, \tag{4.104}$$

is valid. Here the subscripts "m.s" and "n.m" correspond to the methods of

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[23] Data of this section pertain to the basic (unmodified) variations of Newton's method and method of secants.

[24] Data on comparative effectiveness pertain to use of basic (unmodified) versions of the methods for solving equations with computable functions f(x).

secants and Newton's method. Thus the method of secants has better computational effectiveness than Newton's method.

TABLE 4.16. COMPARISON OF COMPUTATIONAL EFFECTIVENESS OF METHOD OF SECANTS AND NEWTON'S METHOD

n	1	2	3	4	6	8	10	20	50
$E_{n.m}$	1.414	1.260	1.190	1.148	1.104	1.080	1.065	1.034	1.014
$E_{m.s}$	1.618	1.466	1.380	1.325	1.255	1.213	1.184	1.114	1.058
R_n	1.388	1.654	1.860	2.028	2.297	2.509	2.684	3.283	4.179

The ratio of the summary number $\theta_{\Sigma_{n.m}}$ of computations of the function f(x) in Newton's method to the summary number $\theta_{\Sigma_{m.s}}$ of computations in the method of secants for a given reduction of error $\varepsilon_{n.m} = \varepsilon_{m.s}$ is [see (4.75)]:

$$\frac{\theta_{\Sigma n.m}}{\theta_{\Sigma m.s}} = R(n) = \frac{\theta_{n.m} \log v_{m.s}(n)}{\theta_{n.m} \log v_{n.m}} = \frac{(n+1)\log v_{m.s}(n)}{\log 2}. \qquad (4.105)$$

R(n) shows, in some respects, how many times more effective the method of secants is than Newton's method (see Table 4.16).

Advantages and disadvantages of method. 1. The method is characterized by the highest computational effectiveness of all known methods of solving equation systems (4.87). /331

2. The method requires a larger volume of stored data (storage of (n + 1) points $x^{(p+r)}$ and (n + 1) vectors $f(x^{(p+r)})$) compared with single point methods (of Newton's type).

3. The method is not self-correcting, i.e., correction of the results of an incorrect preceding step is difficult.

Example 9. Calculation of the parameters of equilibrium balance mode of straight-line flight by method of secants. The nonlinear equation system that describes the mode of straight-line balance flight is (see example 8, Section 4.2.3):

$$f_1(P,\ \alpha) \triangleq P\cos(\alpha+\varphi_p) - \overline{B}_\alpha q_0 S\alpha^2 - c_{x0}q_0 S - mg(h_0)\sin\theta_0 = 0;$$

$$f_2(P,\ \alpha) \triangleq P\sin(\alpha+\varphi_p) + c_y^\alpha q_0 S\alpha - m_0\left[g(h_0) - \frac{v_0^2}{R_3+h_0}\right]\cos\theta_0 = 0.$$

Let be given in some form three initial approximations: $(P^{(-2)},\ \alpha^{(-2)})^T$, $(P^{(-1)},\ \alpha^{(-1)})^T$, $(P^{(0)},\ \alpha^{(0)})^T$ and let the vectors $f = (f_1(P,\ \alpha),\ f_2(P,\ \alpha))^T$ corresponding to these approximations be computed:

$$f^{(-2)} = \begin{bmatrix} f_1(P^{(-2)},\ \alpha^{(-2)}) \\ f_2(P^{(-2)},\ \alpha^{(-2)}) \end{bmatrix};\quad f^{(-1)} = \begin{bmatrix} f_1(P^{(-1)},\ \alpha^{(-1)}) \\ f_2(P^{(-1)},\ \alpha^{(-1)}) \end{bmatrix};$$

$$f^{(0)} = \begin{bmatrix} f_1(P^{(0)},\ \alpha^{(0)}) \\ f_2(P^{(0)},\ \alpha^{(0)}) \end{bmatrix}.$$

Then, in accordance with the computational formulation of the method of secants, the following approximation $(P^{(1)},\ \alpha^{(1)})^T$ is found according to equations (4.89):

$$P^{(1)} = \mu_{(-2)}P^{(-2)} + \mu_{(-1)}P^{(-1)} + \mu_0 P^{(0)};$$

$$\alpha^{(1)} = \mu_{(-2)}\alpha^{(-2)} + \mu_{(-1)}\alpha^{(-1)} + \mu_{(0)}\alpha^{(0)},$$

where $\mu_{(p)}$, (p = -2; - 1; 0) are determined from the solution of equation system (4.90), linear relative to $\mu_{(p)}$:

$$\mu_{(-2)}f_1(P^{(-2)},\ \alpha^{(-2)}) + \mu_{(-1)}f_1(P^{(-1)},\ \alpha^{(-1)}) + \mu_{(0)}f_1(P^{(0)},\ \alpha^{(0)}) = 0;$$

$$\mu_{(-2)}f_2(P^{(-2)},\ \alpha^{(-2)}) + \mu_{(-1)}f_2(P^{(-1)},\ \alpha^{(-1)}) + \mu_{(0)}f_2(P^{(0)},\ \alpha^{(0)}) = 0;$$

$$\mu_{(-2)} + \mu_{(-1)} + \mu_{(0)} = 1.$$

The solution of the linear equation system for $\mu_{(p)}$ for a large number of unknowns is conveniently found by one of the exclusion methods (for instance by the Gauss method) or by the supplement method. When there are few unknowns (n = 2-3) the solution can be found by Cramer's method through determinants.

The subsequent approximations $P^{(r+1)},\ \alpha^{(r+1)})^T$ (r = 1, 2, ...) are found through equations of the form

$$P^{(r+1)} = \mu_{(r-2)}P^{(r-2)} + \mu_{(r-1)}P^{(r-1)} + \mu_{(r)}P^{(r)};$$

$$\alpha^{(r+1)} = \mu_{(r-2)}\alpha^{(r-2)} + \mu_{(r-1)}\alpha^{(r-1)} + \mu_{(r)}\alpha^{(r)},$$

where the variables $\mu_{(p+r)}$ (p = 0, - 1, - 2) are determined from the solution of the system /332

$$\mu_{(r-2)} f_1(P^{(r-2)}, \alpha^{(r-2)}) + \mu_{(r-1)} f_1(P^{(r-1)}, \alpha^{(r-1)}) + \mu_{(r)} f_1(P^{(r)}, \alpha^{(r)}) = 0;$$

$$\mu_{(r-2)} f_2(P^{(r-2)}, \alpha^{(r-2)}) + \mu_{(r-1)} f_2(P^{(r-1)}, \alpha^{(r-1)}) + \mu_{(r)} f_2(P^{(r)}, \alpha^{(r)}) = 0;$$

$$\mu_{(r-2)} + \mu_{(r-1)} + \mu_{(r)} = 1 \quad (r = 0, 1, 2, \ldots).$$

Here the replacement of "old" points $(P^{(p+r)}, \alpha^{(p+r)}$ by the (r + 1)-th iteration can be done either according to the least index p (i.e., cyclically), or according to the maximum discrepancy norm:

$$\max_{-2+r<p+r<r} \left[f_1^2(P^{(p+r)}, \alpha^{(p+r)}) + f_2^2(P^{(p+r)}, \alpha^{(p+r)}) \right].$$

The results of determination of the parameters of steady state flight of an aircraft, numerical aerodynamic characteristics and motion parameters of which are presented in example 8 of Section 4.2.3, are listed in Tables 4.17 a and 4.17 b. The calculation was done for the same two variations of initial conditions as in example 8 of Section 4.2.3. It is obvious that to obtain the solution with an accuracy up to four significant figures it is necessary to compute 5 to 7 times the vector $f(P, \alpha) = (f_1)P, \alpha), f_2(P, \alpha)^T$.

4.3. Numerical Methods of Optimizing Trajectories and Aircraft Flight Controls

The necessary conditions of optimality of control u(t) and of trajectory x(t) corresponding to it are discussed in Chapter 3. The boundary value problems that arise when using necessary conditions of optimality in the form of the maximum principle can be reduced to the solution of nonlinear transcendental equation systems relative to unknown initial conjugate variables λ_{i0} (see Section 4.2). These equations can be solved by numerical methods discussed in section 4.2. Presented below are numerical methods of solving problems of optimal control, not directly related to the use of necessary optimality conditions.

4.3.1. Functional Gradient Method

The method is used for solving optimal programmed control problems.

Initial data. Let controlled motion be described by an ordinary differential equation system:

- in scalar form

$$x_i = f_i(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \quad (i = 1, 2, 3, \ldots, n), \tag{4.106}$$

TABLE 4.17 a. CALCULATION OF PARAMETERS OF EQUILIBRIUM FLIGHT MODE OF AIRCRAFT BY METHOD OF SECANTS USING HIGH SPEED DIGITAL COMPUTER

The procedure of separating the point $(P^{(r-2)}, \alpha^{(r-2)})$, $r \geqslant 0$ by the $(r+1)$-th iteration is used; the deficient initial points $(P^{(-1)}, \alpha^{(-1)})$ and $(P^{(0)}, \alpha^{(0)})$ are found from the point $(P^{(-2)}, \alpha^{(-2)})$ as follows: $(P^{(-1)}, \alpha^{(-1)}) = (P^{(-2)} + \Delta P, \alpha^{(-2)})$, $(P^{(0)}, \alpha^{(0)}) = (P^{(-2)}, \alpha^{(-2)} + \Delta\alpha$, where $\Delta P = 100$ kg, $\Delta\alpha = 0.1°$
(Presented in the table are results printed out by the computer without processing)

Variation of initial Conditions	No. of Iteration r	$P^{(r)}$ kg	$\alpha^{(r)}$ deg.	$f_1(P^{(r)}, \alpha^{(r)})$ kg	$f_2(P^{r)}, \alpha^{(r)})$ kg	$\|\|f(P^{(r)}, \alpha^{(r)})\|\|$ kg	Total number of computations of vector f for r iterations.
1	−2	0	0	−5273,93121	−15043,7867	15941,4513	
	−1	100,000000	0	−5174,06825	−15038,5531	15903,7436	
	0	0	0,100000000	−5275,16819	−14297,7461	15239,8472	3
	1	5305,68561	1,97926349	−472,854079	182,829877	506,969175	4
	2	5810,10872	1,94959886	44,3425581	2,30434918	44,4023928	5
	3	5766,17678	1,94977076	0,490032911	−0,186420679	0,524294692	6
	4	5765,69985	1,94980089	$0{,}896453858 \cdot 10^{-4}$	$0{,}240802765 \cdot 10^{-3}$	$0{,}256947984 \cdot 10^{-3}$	7
	5	5765,69975	1,94980086	$0{,}119209289 \cdot 10^{-6}$	0	$0{,}119209289 \cdot 10^{-6}$	8
	6	5765,69975	1,94980086	0	0	0	9
2	−2	5799,12677	2,01648373	0	507,089209	507,089209	
	−1	5899,12677	2,01648373	99,6169582	515,833443	525,364330	
	0	5799,12677	2,11648373	−52,0179036	1263,211161	1264,28218	3
	1	5764,31732	1,94982186	−1,38757938	$0{,}394873619 \cdot 10^{-1}$	1,38814112	4
	2	5765,68989	1,94980079	$-0{,}978261233 \cdot 10^{-2}$	$-0{,}131416321 \cdot 10^{-2}$	$0{,}987048777 \cdot 10^{-2}$	5
	3	5765,69975	1,94980086	$0{,}441074371 \cdot 10^{-5}$	$0{,}238418579 \cdot 10^{-6}$	$0{,}441718276 \cdot 10^{-5}$	6
	4	5765,69975	1,94980086	0	0	0	7

Note: Commas indicate decimal points.

TABLE 4.17 b. CALCULATION OF PARAMETERS OF EQUILIBRIUM FLIGHT MODE OF AIRCRAFT BY METHOD OF SECANTS USING HIGH SPEED DIGITAL COMPUTER

The procedure of separation of the point with maximum discrepancy mode by the $(r + 1)$-th iteration is used; the deficient initial points $(P^{(-1)}, \alpha^{(-1)})$ and $(P^{(0)}, \alpha^{(0)})$ are found from the point $(P^{(-2)}, \alpha^{(-2)})$ as follows: $(P^{(-1)}, \alpha^{(-1)}) = (P^{(-2)} + \Delta P, \alpha^{(-2)})$; $(P^{(0)}, \alpha^{(0)}) = (P^{(-2)}, \alpha^{(-2)} + \Delta\alpha)$, where $\Delta P = 100$ kg; $\Delta\alpha = 0.1°$.

(Presented in the table are results printed out by the computer without processing.)

Variation of initial Conditions	No. of Iteration r	$P^{(r)}$ kg	$\alpha^{(r)}$ deg.	$f_1(P^{(r)}, \alpha^{(r)})$ kg	$f_2(P^{(r)}, \alpha^{(r)})$ kg	$\|\| f(P^{(r)}, \alpha^{(r)}) \|\|$ kg	Total number of computations of vector f for r iterations.
1	−2	0	0	−5273,93121	−15043,7867	15941,4513	
	−1	100,000000	0	−5174,06825	−15038,5531	15903,7436	
	0	0	0,100000000	−5275,16819	−14297,7461	15239,8472	3
	1	5305,68561	1,97926349	−472,854079	182,829877	506,969175	4
	2	5810,10872	1,94959886	44,3425581	2,30434918	44,4023928	5
	3	5766,17678	1,94977076	0,490032911	−0,186420679	0,524294692	6
	4	5765,69985	1,94980089	$0.896453858 \cdot 10^{-4}$	$-0,240802765 \cdot 10^{-3}$	$0,256947984 \cdot 10^{-3}$	7
	5	5765,69975	1,94980086	$0,119209289 \cdot 10^{-6}$	0	$0,119209289 \cdot 10^{-6}$	8
	6	5765,69975	1,94980086	0	0	0	9
2	−2	5799,12677	2,01648373	0	507,089209	507,089209	
	−1	5899,12677	2,01648373	99,6169582	515,833443	525,364330	
	0	5799,12677	2,11648373	−52,0179036	1263,211161	1264,28218	3
	1	5764,31732	1,94982186	−1,38757938	$0,394873619 \cdot 10^{-1}$	1,38814112	4
	2	5765,69961	1,94980065	$-0,363588333 \cdot 10^{-4}$	$-0,160264969 \cdot 10^{-2}$	$0,160306206 \cdot 10^{-2}$	5
	3	5765,69975	1,94980086	$0,178813934 \cdot 10^{-5}$	0	$0,178813934 \cdot 10^{-5}$	6
	4	5765,69975	1,94980086	0	$0,238418579 \cdot 10^{-6}$	$0,238418579 \cdot 10^{-6}$	7
	5	5765,69975	1,94980086	0	0	0	8

Note: Commas indicate decimal points.

- in vectorial form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \tag{4.107}$$

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where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the phase coordinate vector;

$\mathbf{u} = (u_1, u_2, \ldots, u_m)$ is the control vector.

Here control u is an element of some open region U^m:

$$\mathbf{u} \in U^m.$$

Initial conditions. Given for $t = t_0$ is the vector x:

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{4.108}$$

Boundary conditions. At the final moment of time $t = t_1$, which is not given, the condition

$$\varphi(t_1, \mathbf{x}(t_1)) = 0. \tag{4.109}$$

is satisfied.

Optimality criterion. The functional J[u] to be optimized is given (for definition, the minimization functional):

$$J[\mathbf{u}] = g_0(t_1, \mathbf{x}(t_1)) + \int_{t_0}^{t_1} f_0(t, \mathbf{x}, \mathbf{u})\, dt. \tag{4.110}$$

Required: to find control vector

$$\mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_m(t)), \tag{4.111}$$

such that the solution of system (4.107) satisfies the initial and boundary conditions (4.108), (4.109) and functional (4.110) attains its minimum.

Idea of method. Introduced are the functions

$$H = \lambda_0 f_0(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1} \lambda_i f_i(t, \mathbf{x}, \mathbf{u}); \tag{4.112}$$

$$G = \lambda_0 g_0(t_1, \mathbf{x}(t_1)) + \mu\varphi(t_1, \mathbf{x}(t_1)), \tag{4.113}$$

where $\lambda_0 = 1$;

$\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the vector of conjugate variables;

μ is a constant Lagrange factor.

The vector λ satisfies the equation

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}. \tag{4.114}$$

For variation δJ of functional $J[u]$ at moment of time $t = t_1$, when condition $\phi(t_1, x(t_1)) = 0$ is satisfied, the following expression is valid:

$$\delta J|_{t=t_1} = H\delta t - \sum_{i=1}^{n} \lambda_i \delta x_i \Big|_{t_0}^{t_1} + dG + \sum_{i=1}^{m} \int_{t_0}^{t_1} \frac{\partial H}{\partial u_i} \delta u_i \, dt. \tag{4.115}$$

Now $\lambda_i(t_1)$ and μ are selected such that for all δt_1, $\delta x_i(t_1)$ the condition /336

$$H\delta t + \sum_{i=1}^{n} \lambda_i \delta x_i \Big|_{t_0}^{t_1} + dG = H\delta t - \sum_{i=1}^{n} \lambda_i \delta x_i \Big|_{t_0}^{t_1} + \frac{\partial g_0}{\partial t_1} \delta t_1 +$$

$$+ \sum_{i=1}^{n} \frac{\partial g_0}{\partial x_i(t_1)} \delta x_i(t_1) + \mu \left(\frac{\partial \varphi}{\partial t_1} \delta t_1 + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i(t_1)} \delta x_i(t_1) \right) = 0, \tag{4.116}$$

i.e.,

$$f_0 + \sum_{i=1}^{n} \lambda_i(t_1) \dot{x}_i(t_1) + \frac{\partial g_0}{\partial t_1} + \mu \frac{\partial \varphi}{\partial t_1} = 0; \tag{4.117}$$

$$-\lambda_i(t_1) + \frac{\partial g_0}{\partial x_i(t_1)} + \mu \frac{\partial \varphi}{\partial x_i(t_1)} = 0 \tag{4.118}$$

is satisfied. Hence

$$\lambda_i(t_1) = \frac{\partial g_0}{\partial x_i(t_1)} + \mu \frac{\partial \varphi}{\partial x_i(t_1)}; \tag{4.119}$$

$$\mu = -\frac{\dot{g}_0 + f_0}{\dot{\varphi}}\Big|_{t=t_1}, \tag{4.120}$$

where the complete derivatives with respect to time $\dot{g}_0(t_1)$ and $\dot{\phi}(t_1)$ are determined by numerical integration of system (4.107).

Then expression (4.115) becomes

$$\delta J|_{t=t_1} = \int_{t_0}^{t_1} \sum_{i=1}^{m} \frac{\partial H}{\partial u_i} \delta u_i \, dt. \tag{4.121}$$

If we select

$$\delta u_i = k^i \frac{\partial H}{\partial u_i}, \tag{4.122}$$

where k^i is the weight coefficient, then the increment δJ of the functional is

$$\delta J|_{t=t_1} = \int_{t_0}^{t_1} \sum_{i=1}^{m} k^i \left(\frac{\partial H}{\partial u_i}\right)^2 dt. \qquad (4.123)$$

Consequently the increment δJ of the functional $J[u(t)]$ on variations δu_i has a fixed sign:

for $k^i > 0$ $\delta J \geqslant 0$ (maximization);

for $k^i < 0$ $\delta J \leqslant 0$ (minimization).

Using variation δJ of fixed sign it is possible to construct the sequence of controls $\{u^{(r)}(t)\}$, to which corresponds the diminishing sequence of values of the functional

$$J[u^{(0)}(t)]|_{t_1^{(0)}} > J[u^{(1)}(t)]|_{t_1^{(1)}} > \ldots > J[u^{(r)}(t)]|_{t_1^{(r)}} > \ldots \qquad (4.124)$$

If the functional $J[u(t)]$ is bounded below, then the process of successive approximations leads not only to a convergent sequence of values $J[u^{(r)}(t)]$, but also to a sequence of functions convergent on zero on the segment $[t_0, t_1]$: /337

$$\{\delta u^{(r)}(t)\} \qquad (r=0, 1, 2, \ldots). \qquad (4.125)$$

Computational structure. The algorithm of solution is the following.

1. From any considerations the initial approximation of vector $u(t)$ is selected:

$$u^{(0)}(t) = (u_1^{(0)}(t), u_2^{(0)}(t), \ldots, u_m^{(0)}(t)). \qquad (4.126)$$

2. For given vector x_0 on segment $t_0 \leqslant t \leqslant t_1$ system (4.107) with control $u^{(0)}(t)$ is integrated until the condition $\phi(t_1, x(t_1)) = 0$ is satisfied. For this moment of time are numerically or analytically determined $x(t_1)$, $\frac{\partial g_0}{\partial x_i(t_1)}$, $\frac{\partial \varphi}{\partial x_i(t_1)}$ and $\left.\frac{dg_0}{dt}\right|_{t=t_1}$ and $\left.\frac{d\varphi}{dt}\right|_{t=t_1}$ are numerically determined. The value t_1 is found by interpolation from the condition (4.109).

3. The vector λ and number μ are computed at the point $t = t_1$ [see (4.119), (4.120)].

4. The weight coefficients k^i are designated and the basic and conjugate systems (4.107) and (4.114) are integrated for $u^{(0)}(t)$ in "reverse" time

$\tau = t_1 - t$ for initial conditions:

$$\tau=0,\ x(0)=x(t_1),\ \lambda(0)=\lambda(t_1). \tag{4.127}$$

During the integration process the function $\lambda(t)$ and $\delta u_i^{(0)}(t)=k^l\frac{\partial H(u^{(0)})}{\partial u_i}$ are computed and the values of

$$u_i^{(1)}(t)=u_i^{(0)}+\delta u_i^{(0)}. \tag{4.128}$$

are determined and stored in the memory.

The latter expression for $u_i^{(1)}$ is as a new (improved) control.

5. Instructions 1-4 are repeated until one of the assumed conditions of completion of the computation is satisfied, for instance until $\frac{\partial H}{\partial u_i}$ or $\int_{t_0}^{t_1}\frac{\partial H}{\partial u}\left(\frac{\partial H}{\partial u}\right)^{\tau}dt$ are small or until the differences

$$|J[u^{(r+1)}]-J[u^{(r)}]|\leqslant\varepsilon_J \tag{4.129}$$

are less than a given quantity;

$$\max_{0\leqslant t\leqslant t_1}|u_l^{(r+1)}(t)-u_l^{(r)}(t)|\leqslant\varepsilon_{ul}\quad(r=0,1,2,\ldots;\quad l=1,2,\ldots,m). \tag{4.130}$$ /338

Comment. These conditions theoretically are not a proved condition of completion of computations. Only in quite "good" cases can checking of one of them (preferably several) be a reliable practical indication of finding the result with the required accuracy.

Choice of weight coefficients k^i. It is often difficult in specific problems to make an intelligent choice of k^i. One means of selection consists in designating k^i by the scheme

$$k^l=\begin{cases}-1, & \text{if}\quad \delta u_{i\,\mathrm{kn}}^{\min}\leqslant\delta u_l(t)\leqslant\delta u_{l\,\mathrm{kn}}^{\max};\\ -\dfrac{\delta u_{l\,\mathrm{kn}}^{\max}}{\left|\max\limits_{0\leqslant t\leqslant t_1}\delta u_l(t)\right|}, & \text{if}\quad \left|\max\limits_{0\leqslant t\leqslant t_1}\delta u_l(t)\right|>\delta u_{l\,\mathrm{kn}}^{\max}\\ & \text{or}\quad \left|\max\limits_{0\leqslant t\leqslant t_1}\delta u_l(t)\right|<\delta u_{l\,\mathrm{kn}}^{\min}\end{cases} \tag{4.131}$$

or:

$$k^l=k=\arg\min_{k>0}J[u^{(r)}(t)+k\delta u^{(r)}(t)];\quad \delta u^{(r)}=\frac{\partial H(u^{(r)})}{\partial u}. \tag{4.132}$$

Rate of convergence of method. Gradient methods have only linear rates of convergence. Therefore they often converge extremely slowly, particularly in the vicinity of the solution. The general recommendation concerning their use is that they should be used in the initial stages of improvement of control.

4.3.2. Method of Penalty Functions

The method of penalty functions makes it possible to solve problems with several boundary conditions.

Initial data. Let controlled motion be described by a system of ordinary differential equations:

- in scalar form

$$x_i = f_i(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \quad (i = 1, 2, \ldots, n); \tag{4.133}$$

- in vectorial form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \tag{4.134}$$

where $x = (x_1, x_2, \ldots, x_n)^T$ is the phase coordinate vector;
$u = (u_1, u_2, \ldots, u_m)^T$ is the control vector;
$f = (f_1, f_2, \ldots, f_n)^T$.

Then control u is an element of some open region U: /339

$$\mathbf{u} \in U.$$

Initial conditions. Given for $t = t_0$ is the vector x:

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{4.135}$$

Boundary conditions. At the final moment of time $t = t_1$, which is not known, are satisfied the condition $\phi(t_1, x(t_1)) = 0$ and $q \leqslant n$ conditions of the form

$$\varphi_j(t_1, \mathbf{x}(t_1)) = 0 \qquad (j = 1, 2, \ldots, q). \tag{4.136}$$

Optimality criterion. The functional J[u] to be optimized is given (for definition, the minimization functional):

$$J[\mathbf{u}] = g_0(t_1, \mathbf{x}(t_1)) + \int_{t_0}^{t_1} f_0(t, \mathbf{x}, \mathbf{u})\, dt. \tag{4.137}$$

Required: to find the control vector

$$\mathbf{u}(t)=(u_1(t),\ u_2(t),\ \ldots,\ u_m(t)), \tag{4.138}$$

for which the solution of system (4.134) satisfies the initial and boundary conditions (4.135) and (4.136) and minimizes the functional (4.137).

The idea of the method consists in using the computational structure of the functional gradient method (see Section 4.3.1), but examining, instead of the given functional (4.137), the auxiliary functional

$$J_1[\mathbf{u}]=J[\mathbf{u}]+\Phi_1(\varphi_1,\ \ldots,\ \varphi_q), \tag{4.139}$$

where Φ_1 is the measure of deviation from the prescribed boundary conditions (penalty function).

Computational structure. The computation corresponds completely to the system used in Section 4.3.1, where $J[u]$ is replaced by $J_1[u]$. Quite often the function Φ_1 is taken in the form

$$\Phi_1=\Phi_1(t_1,\ \mathbf{x}(t_1))=\sum_{j=1}^{q}\nu_j\varphi_j^2(t_1,\ \mathbf{x}(t_1)), \tag{4.140}$$

where $\nu_j > 0$ are weight factors.

Comments on the use of the method. 1. It is known that the solution of the problem of unconditional minimization of functional $J_1[u]$ approaches the solution of the problem of minimization of $J[u]$ in the presence of constraints $\Phi_j = 0$ if $\nu_j \to \infty$. In practice it is sometimes possible to choose the final values of ν_j for which the boundary conditions are satisfied with the prescribed accuracy. However, the introduction of "penalty" terms often interferes with the process of convergence with respect to the basic functional $J[u]$ and control $u(t)$. /340

2. The penalty functions method can also be used for a problem with limitations of the inequality type:

$$\psi_l(t,\ \mathbf{x},\ \mathbf{u})\geqslant 0 \qquad (l=1,\ 2,\ 3,\ \ldots,\ s). \tag{4.141}$$

Introduced in this case is the function

$$\Phi_2(t_1,\ \mathbf{x}(t_1))=\sum_{l=n+1}^{n+s}\gamma_l x_l(t_1), \tag{4.142}$$

where $\gamma_i > 0$, a $x_l(t)$ $(l=n+1,\ n+2,\ \ldots,\ n+s)$

satisfies the system

$$x_l = \exp\{-\psi_l(t, \mathbf{x}, \mathbf{u})\} \tag{4.143}$$

with zero initial conditions.

This system is connected with the basic system and the functional gradient method (see Section 4.3.1) is used for minimizing the auxiliary functional

$$J_2[\mathbf{u}] = J[\mathbf{u}] + \Phi_2. \tag{4.144}$$

The process of convergence may proceed very slowly.

The methods described above should be used in the first stage of computation, when control and boundary conditions are far from optimal.

4.3.3. Newton-Kantorovich Method[25]

This method is used for solving two-point boundary value problems of optimal control. It is a generalization for the case of the boundary value problem for th system of ordinary differential equations of the Newton method for solving nonlinear algebraic and transcendental equation systems (see Section 4.2.3).

Initial data. Let be given the differential equation system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \lambda} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}); \tag{4.145}$$

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial \mathbf{x}}, \tag{4.146}$$

where x, f, λ are n-dimensional vectors; u is an m-dimensional vector; /341

$$H = \sum_{l=1}^{n} \lambda_l f_l(t, \mathbf{x}, \mathbf{u}) + f_0(t, \mathbf{x}, \mathbf{u}); \tag{4.147}$$

$$\mathbf{u} = \mathbf{u}(t, \mathbf{x}, \lambda) = \arg\min_{u \in U} H(t, \mathbf{x}, \mathbf{u}); \tag{4.148}$$

$$t \in [t_0, t_1]; \quad x \in X; \quad \lambda \in \Lambda; \quad \mathbf{u} \in U;$$

X, Λ, U are some open sets.

Boundary conditions. Given for $t = t_0$ and $t = t_1$ are the following conditions:

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[25] Also known as the quasilinearization method.

$$x_l(t_0)=x_{l0}; \tag{4.149}$$

$$x_l(t_1)=x_{l1} \qquad (l=1, 2, \ldots, n). \tag{4.150}$$

Required: to find $x(t)$ and $\lambda(t)$ satisfying systems (4.145), (4.146) and boundary conditions (4.149) and (4.150).

The idea of the method is that if initial approximations $x^{(0)}(t)$ and $\lambda^{(0)}(t)$ are given, then the following approximations $x^{(r+1)}(t)$ and $\lambda^{(r+1)}(t)$ are found from a system of differential equations, linear relative to them, obtained by linearization of system (4.145)-(4.146) relative to the values $\Delta x = x^{(r+1)} - x^{(r)}$ and $\Delta\lambda = \lambda^{(r+1)} - \lambda^{(r)}$:

$$\dot{x}^{(r+1)}=H_{\lambda x}(t, x^{(r)}, u^{(r)})(x^{(r+1)}-x^{(r)})+H_{\lambda u}(t, x^{(r)}, u^{(r)})\times$$
$$\times(u^{(r+1)}-u^{(r)})+f(t, x^{(r)}, u^{(r)}); \tag{4.151}$$

$$\dot{\lambda}^{(r+1)}=-H_{xx}(t, x^{(r)}, u^{(r)})(x^{(r+1)}-x^{(r)})-H_{xu}(t, x^{(r)}, u^{(r)})\times$$
$$\times(\lambda^{(r+1)}-\lambda^{(r)})-H_{xu}(t, x^{(r)}, u^{(r)})(u^{(r+1)}-u^{(r)})-\frac{\partial H}{\partial x}; \tag{4.152}$$

$$u^{(r+1)}=u^{(r)}-H_{uu}(t, x^{(r)}, u^{(r)})\left[H_{ux}(t, x^{(r)}, u^{(r)})(x^{(r+1)}-x^{(r)})+\right.$$
$$\left.+H_{u\lambda}(t, x^{(r)}, u^{(r)})(\lambda^{(r+1)}-\lambda^{(r)})\right], \tag{4.153}$$

where $H_{\lambda x}, H_{\lambda u}, H_{xx}, H_{uu}, H_{ux}, H_{u\lambda}$ are the matrices of the partial derivatives in terms of (λ, x), (λ, u), (x, x), (u, u), (u, x), (u, λ) (i.e., $\frac{\partial^2 H}{\partial\lambda\,\partial x}$, $\frac{\partial^2 H}{\partial\lambda\,\partial u}$, $\frac{\partial^2 H}{\partial x^2}$ etc.) computed on the trajectory of the r-th approximation.

Here the following boundary conditions must be satisfied:

$$x_l^{(r+1)}(t_0)=x_{l0}; \tag{4.149'}$$

$$x_l^{(r+1)}(t_1)=x_{l1}. \tag{4.150'}$$

Under certain conditions the sequence $\{x^{(r)}, \lambda^{(r)}\}$ converges on the solution (x, λ^*) of the initial boundary problem.

Computational structure. In each $(r+1)$-th iteration step must be solved the boundary value problem for system (4.151)-(4.153) with conditions (4.149'), (4.150'). /342

Writing this system and the boundary conditions in vectorial form

$$\dot{z}=G_1(t)z(t)+D(t); \tag{4.154}$$

$$z_i(t_0)=y_{i0}; \tag{4.155}$$

$$z_i(t_1)=y_{i1}, \quad (i=1, 2, \ldots, n) \tag{4.156}$$

where

$$\mathbf{z}=(y_1, y_2, \ldots, y_N); \tag{4.157}$$
$$N=2n,$$

we obtain the following algorithm of solution of this linear boundary value problem.

1. The set of solutions $\{\mathbf{z}^{(n+i)}(t)\}$ (i = 1, 2, ..., n) for homogeneous system

$$\dot{\mathbf{z}}=G_1(t)\mathbf{z} \tag{4.158}$$

is determined, with initial conditions

$$\mathbf{z}^{n+i}(t_0)=(0, 0, \ldots, z_{n+i}=1, 0, \ldots, 0) \quad (i=1, 2, \ldots, n). \tag{4.159}$$

2. The partial solution of inhomogeneous system (4.154) with initial conditions of the form

$$\bar{\mathbf{z}}(t_0)=(y_{10}, y_{20}, \ldots, y_{n0}, k_1, \ldots, k_n), \tag{4.160}$$

is found, where k_j (j = 1, 2, ..., n) are arbitrary numbers, zeroes for example.

3. The general solution of system (4.154) is written:

$$\mathbf{z}(t)=c_{n+1}\mathbf{z}^{(n+1)}(t)+c_{n+2}\mathbf{z}^{n+2}(t)+ \\ +\ldots+c_N\mathbf{z}^N(t)+\bar{\mathbf{z}}(t). \tag{4.161}$$

4. The constants $c_{(n+i)}$ (i = 1, 2, ..., n) are determined from the second system of boundary conditions (4.150').

Comment. To conserve the operational memory of the computer and to check the accuracy of solution of the linear boundary value problem it is advisable to find z(t) not from system (4.161), but by numerical integration of system (4.154) for initial conditions

$$\mathbf{z}(t_0)=(y_{10}, y_{20}, \ldots, y_{n0}, c_{n+1}+k_1, c_{n+2}+k_2, \ldots, c_N+k_n). \tag{4.162}$$

In this case it is not necessary to store the vectors $\mathbf{z}^{n+i}(t)$ and $\bar{\mathbf{z}}(t)$, but only their values at the final moment of time $t = t_1$. The vector z(t) should be stored in the computer, since it is used for computing the matrices of partial derivatives $H_{\lambda x}$, etc., and the right sides of the system in the next iteration step.

5. If time t_1 is not known, then the following method can be used. Certain initial approximations are given for the time of completion of integration $t_1 = t_1^{(0)}$, and one of the components of the vector $x(t_1)$, $x_j(t_1)$, for example, is left free (i.e., is not inserted in the boundary conditions of the boundary value problem). Then the problem is solved by the same method for fixed time $t_1 = t_1^{(0)}$. The times $t_1 = t_1^{(1)}$ are reassigned and the calculation is repeated. Then time t_1 is corrected by means of linear interpolation to the given value $x_j(t_1) = x_{j1}$: /343

$$t_1^{(2)} = t_1^{(1)} + \frac{t_1^{(1)} - t_1^{(0)}}{x_j(t_1^{(1)}) - x_j(t_1^{(0)})} (x_{j1} - x_j(t_1^{(1)})). \tag{4.163}$$

If need be, calculation by equation (4.163) is repeated until the condition

$$\tilde{\varrho} \leqslant \varepsilon,$$

is satisfied, where

$$\tilde{\varrho} = \varrho + \frac{1}{b} |t_1^{(r+1)} - t_1^{(r)}|,$$

$$\varrho = \sum_{i=1}^{n} \max_{t \in [t_0, t_1]} |x_i(t^{(r+1)}) - x_i(t^{(r)})| +$$

$$+ \sum_{i=1}^{n} \max_{t \in [t_0, t_1]} |\lambda_i(t_1^{(r+1)}) - \lambda_i(t_1^{(r)})|,$$

and b is the scale coefficient.

Convergence of method. The method has quadratic convergence. In the case of extremely rough initial approximations the method may have to be modified in order to eliminate divergence.

Advantages and disadvantages of method.

1. The method recommended for use in cases when:

- it is easy to specify a sufficiently close initial approximation in the form $\{x^{(0)}(t), \lambda^{(0)}(t)\}$;

- a series of similar variation problems is solved, for which a typical solution is already known; then the solution of the typical problem can be used as the initial approximations for subsequent problems.

2. The functions of the initial approximation can be chosen piece-wise smooth; the control variables can even be discontinuous.

3. Initial approximations do not necessarily satisfy the boundary conditions.

4. The method does not guarantee optimality of the solution as a whole, but does satisfy the necessary conditions of optimality.

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